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On Asymmetric Structures of FGM Copulas: Measures of Association and Properties

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Abstract

This paper introduces a generalization of the asymmetric Farlie–Gumbel–Morgenstern (FGM) copula family, along with several new extensions. General formulas for association measures of these copulas are derived. Necessary and sufficient conditions for certain dependence concepts within this family are also investigated. Since the original FGM copula has a limited correlation range, extending this domain is essential. The proposed generalized FGM family successfully improves the correlation coefficient, offering a broader range of dependence modeling. This key feature enhances the family's applicability across various scientific disciplines.

Keywords: Farlie–Gumbel–Morgenstern copula, Dependence measures, Asymmetric copula, Correlation range.

1 | Introduction

Copulas have become an essential tool in modern statistics for modeling multivariate dependence structures. Formally, a copula is a function that links univariate marginal distribution functions to form a joint multivariate distribution, particularly in the bivariate case. According to Sklar [1], copulas are bivariate distribution functions defined on the unit square $[0,1]^2$ with uniformly distributed margins on $[0,1]$.

Sklar's theorem establishes that for any bivariate distribution function H with continuous marginal distributions $F(X)$ and $G(Y)$, there exists a unique copula C such that $H_\theta(X,Y)=C(F(X),G(Y);\theta)$. This representation provides a flexible and powerful approach to model bivariate distributions. Once the marginal distributions and an appropriate copula are determined, the joint distribution function can be readily constructed (for further details, see Joe [2], Nelsen [3]). Among the numerous parametric families of copulas, the Farlie–Gumbel–Morgenstern (FGM) copula, independently studied by Farlie [4], Gumbel [5], and Morgenstern [6], remains one of the most popular due to its simple analytical form.

The standard FGM copula is defined as $C^{FGM}(u,v)=uv\left[1+\theta(1-u)(1-v)\right]$, where $\theta\in[-1,1]$ is the dependence parameter. Owing to its mathematical tractability, the FGM copula is widely used in modeling and in assessing

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the efficiency of nonparametric procedures. However, it is well known that the FGM copula suffers from a significant limitation: Its dependence range is severely restricted. Specifically, for the dependence parameter $\theta \in [-1,1]$, Spearman's rho is and Kendall's, implying that $\rho_S = \theta/3 \in [-1/3, 1/3]$ and $\tau_k = 2\theta/9 \in [-2/9, 2/9]$.

This narrow range often renders the FGM copula inadequate for modeling real-world data exhibiting moderate or strong dependence. In response to this limitation, several generalizations have been proposed to extend the correlation range of the FGM family. Huang and Kotz [7] introduced polynomial-type single-parameter extensions of the FGM copula, demonstrating that Spearman's rho could be increased up to approximately 0.375, while the lower bound was extended to about −0.33. Subsequently, Bairamov and Kotz [8] further extended this family and studied the associated Spearman's rho. Lai and Xie [9] derived conditions for positive quadrant dependence and generalized the uniform representation of the FGM copula to obtain families of bivariate distributions with prespecified marginals and positive quadrant dependence. An alternative semi-parametric symmetric generalization was proposed by Fischer and Klein [10] and extensively studied by Amblard and Girard [11]. Cuadras and Díaz [12] introduced an extended FGM family in two dimensions and explored the approximation of arbitrary distributions within this family. More recently, Bekrizadeh et al. [13] proposed a new class of generalized FGM copula and showed that their generalization successfully improves the correlation domain compared to the classical FGM copula.

Building upon these developments, this paper proposes a new generalization of the FGM copula. The proposed family not only encompasses several existing extensions introduced in recent years but also achieves a further improved correlation range. In particular, the presented family can be viewed as an asymmetric extension of the generalized FGM copula discussed in Bekrizadeh et al. [13]. The main contributions of this paper are as follows. First, we introduce an asymmetric extension of the FGM copula and investigate its fundamental properties. Second, we study the asymmetry characteristics and derive general formulas for association measures, including Spearman's rho and Kendall's tau, for this family.

The key feature of the proposed family is its ability to model a significantly wider range of dependence compared to the classical FGM copula and many of its existing generalizations. This enhanced flexibility expands the potential applications of the family across various scientific disciplines, including finance, environmental science, biostatistics, and reliability engineering. The remainder of this paper is organized as follows. Section 2 describes the new asymmetric extension and its basic properties. Section 3 provides the derivation of Spearman's rho and Kendall's tau for the proposed family. Section 4 is dedicated to the dependence structure properties, including quadrant dependence and other stochastic ordering concepts. Finally, Section 5 concludes the paper with a summary of the findings.

2 | Methodology and Definition

The copula is mostly defined as a function $C:[0,1]^2 \to [0,1]$ that satisfies the boundary conditions

I. A1. $C(u,0) = C(0,u) = 0$ and $C(u,1) = C(1,u) = u$, $\forall u \in [0,1]$,

II. A2. $\forall (u_1, u_2, v_1, v_2) \in [0,1]^4$, such that $u_1 \le u_2$ and $v_1 \le v_2$, $C(u_2,v_2) - C(u_2,v_1) - C(u_1,v_2) + C(u_1,v_1) \ge 0$.

Eventually, for twice differentiable, 2-increasing *Property (A2)* can be replaced by the condition

$$c(u,v) = \frac{\partial^2}{\partial u \partial v} C(u,v) \ge 0, \tag{2.1}$$

where $c(u,v)$ is the so-called copula density. A copula C is symmetric if $C(u,v)=C(v,u)$, for every $(u,v) \in [0,1]^2$, otherwise C is asymmetric.

Many different copulas can be found in the literature; see for instance Nelson [3]. The FGM copula is one of these copulas that are widely used in applications. The FGM copula is defined by

$$C^{FGM}(u,v)=uv\left[1+\theta(1-u)(1-v)\right],\theta\in[-1,1].$$

These copulas are the only ones whose functional form is quadratic in both u and v. Regarding the limitation of correlation range in FGM copula, along with other generalizations, Bekrizadeh et al. [13] proposed a new class of symmetric generalized FGM family whose dependence is as follows:

$$C(u,v)=uv\left[1+\theta\left(1-u^{\alpha}\right)\left(1-v^{\alpha}\right)\right]^{p},\alpha>0 \text{ and } p=0,1,2,\ldots.$$

In the following definition, an asymmetric extension of the above copula family is introduced in order to extend the FGM copula.

Definition 1. Suppose that the continuous functions $A_1,A_2:[0,1]\to[0,1]$ are differentiable on $(0,1)$. An asymmetric function $C_{\theta,p}^{A_1,A_2}:[0,1]\times[0,1]\to[0,1]$ is defined as

$$C_{\theta,p}^{A_1,A_2}(u,v)=uv\left[1+\theta A_1(u)A_2(v)\right]^{p},p\in[0,\infty),\ \forall(u,v)\in[0,1]^2, \tag{2.2}$$

where the parameter $\theta\in\Theta=[-1,1]$ is called the associated parameter.

We assume that A_1 and A_2 do not change their sign on $[0,1]$ in order to obtain a uniquely determined dependence structure. Note that the copula is limited to the range of $[0,1]$ and therefore, $[1+\theta A_1(u)A_2(v)]^p$ should be bounded on $[0,1]$.

The following theorem gives sufficient and necessary conditions on A_1 and A_2 to ensure that $C_{\theta,p}^{A_1,A_2}$ is a bivariate copula.

Theorem 1. Let $A_1,A_2:[0,1]\to[0,1]$ be continuously differentiable functions on $(0,1)$. The function $C_{\theta,p}^{A_1,A_2}$ is a bivariate copula if and only if A_1 and A_2 satisfy the following conditions:

I. B1. $A_i(1)=0$, for $i=1,2$,

II. B2. $|xA_i'(x)|\le 1$ and $|A_i(x)+pxA_i'(x)|\le 1$, for every $x\in[0,1]$, and for $i=1,2$, where $A_i'(x)=\partial A_i(x)/\partial x$.

Proof: The proof involves two steps:

I. It is clear that $C_{\theta,p}^{A_1,A_2}(x,1)=C_{\theta,p}^{A_1,A_2}(1,x)=x$, $\forall x\in[0,1]\Leftrightarrow$ (B1).

II. Since A_1 and A_2 are continuously differentiable functions and $[1+\theta A_1(u)A_2(v)]^p$ is bounded on $[0,1]$ and 2-increasing function, by *Eq. (2.1),* the condition $c_{\theta,p}^{A_1,A_2}(u,v)\ge 0$ holds, if and only if $|xA_i'(x)|\le 1$ and $|A_i(x)+pxA_i'(x)|\le 1$, for every $x\in[0,1]$, and $i=1,2$, where $c_{\theta,p}^{A_1,A_2}(u,v)$ is as follows:

$$\begin{aligned}c_{\theta,p}^{A_1,A_2}(u,v)&=\partial^2 C_{\theta,p}^{A_1,A_2}(u,v)/\partial u\partial v=[1+\theta A_1(u)A_2(v)]^{p-2}\\&\times\left\{\left(1+\theta A_2(v)\left[A_1(u)+puA_1'(u)\right]\right)\left(1+\theta A_1(u)\left[A_2(v)+pvA_2'(v)\right]\right)+p\theta uA_1'(u)vA_2'(v)\right\}.\blacksquare\end{aligned} \tag{2.3}$$

Note that θ is a parameter that shows the dependence structure of the family $C_{\theta,p}^{A_1,A_2}$ and $\theta=0$ or $p=0$, leads to the independence of u and v. By *Theorem 1*, the concrete amount of the parameter space θ is dependent on the properties of the function of A_1, and A_2 that has been investigated via *Eq. (2.3)* for every amount of u and v in $[0,1]$. If $A_1(x)=A_2(x)$, $\forall x\in[0,1]$, then the family $C_{\theta,p}^{A_1,A_2}$ is symmetric.

Remark 1. The family $C_{\theta,p}^{A_1,A_2}$ includes some known families of FGM copulas introduced by researchers in recent years, which are as follows:

I. $A_i(x)=1-x$, $\forall x\in[0,1]$, for $i=1,2$, and $p=1$, the family $C_{\theta,p}^{A_1,A_2}$ leads to the symmetric FGM copula discussed by Farlie [4], Gumbel [5], and Morgenstern [6].

II. If $A_i(x)=1-x^{\alpha}$, $\forall x\in[0,1]$, for $i=1,2$, $\alpha\geq 0$, and $p=1$, the family $C_{\theta,p}^{A_1,A_2}$ leads to the symmetric extended FGM copula introduced by Huang and Kotz [7].

III. If $A_i(x)=x^q(1-x)^q$, $\forall x\in[0,1]$, for $i=1,2$, $q\geq 1$ and $p=1$, the family $C_{\theta,p}^{A_1,A_2}$ leads to the symmetric extended FGM copula introduced by Lai and Xie [9].

IV. If $A_i(x)=(1-x^{\gamma})^{\lambda}$, $\forall x\in[0,1]$, for $i=1,2$, $\gamma\geq 0$, $\lambda\geq 1$ and $p=1$, the family $C_{\theta,p}^{A_1,A_2}$ leads to the symmetric extended FGM copula introduced by Bairamov-Kotz [8].

V. If $A_i(x)=A(x)$, $\forall x\in[0,1]$, for $i=1,2$, and $p=1$, the family $C_{\theta,p}^{A_1,A_2}$ leads to the symmetric copula introduced by Rodriguez-Lallena and Ubeda-Flores [14].

Moreover, via the family $C_{\theta,p}^{A_1,A_2}$, some new generalizations can be defined by introducing additional parameters p for the families (I)-(V), and we can generate some copulas of the family $C_{\theta,p}^{A_1,A_2}$ of type *Eq. (2.2)* through functions A_1 and A_2.

Proposition 1. Two limited properties of the family $C_{\theta,p}^{A_1,A_2}$ are as follows:

I. $\lim_{p\to 0} C_{\theta,p}^{A_1,A_2}(u,v)=\lim_{p\to 0} uv\left[1+\theta A_1(u)A_2(v)\right]^p=uv=\Pi(u,v)$, where $\Pi(u,v)$ is the independent copula.

II. Let $\theta=\frac{\gamma}{p}$, where $\gamma\leq p$, then

$$\lim_{p\to\infty} C_{\theta,p}^{A_1,A_2}(u,v)=\lim_{p\to\infty} uv\left[1+\frac{\gamma}{p}A_1(u)A_2(v)\right]^p=uv\exp\left[\gamma A_1(u)A_2(v)\right]=C_{\gamma}(u,v).\ \blacksquare \tag{2.4}$$

The new family of copulas in *Eq. (2.4)* can be a new asymmetric generalization of Cuadras copula [15].

3 | Meassures of Dependence

Measures of dependence are common instruments to summarize a complicated dependence structure in the bivariate case. For a historical review of measures of dependence, see Joe [2] and Nelsen [3]. In this section, we compute the measures of dependence for the family $C_{\theta,p}^{A_1,A_2}$. Since we cannot give formulas for the properties of dependence in terms of elementary functions, they are replaced by their expansion series on

$\Omega=\left\{(\theta,A_1,A_2):\ |\theta A_1(u)A_2(v)|<1\right\}.$

Based on Ω, the family $C_{\theta,p}^{A_1,A_2}$ in *Eq. (2.2),* for every $p\in[0,\infty)$ may also be written by polynomial expansion with respect to A_1 and A_2 as

$$C_{\theta,p}^{A_1,A_2}(u,v)=uv+\sum_{k=1}^{g}\binom{p}{k}\theta^k uA_1{}^k(u)vA_2{}^k(v). \tag{3.1}$$

Note that, in *Eq. (3.1)*, we have $g=p$ when p is an integer, otherwise, g equals to $+\infty$.

3.1 | Spearman's Rho

Let X and $Y\sqrt{a^2+b^2}$ be continuous random variables whose copula is C. Then the population version of Spearman's rho for X and Y is given by

$$\rho_S = 12\int_0^1\int_0^1 C(u,v)dudv - 3.$$

Note that ρ_S coincides with the correlation coefficient ρ between the uniform marginal distributions.

Proposition 2. Let (X,Y) be a pair of random variables with the family $C_{\theta,p}^{A_1,A_2}$. The Spearman's rho (ρ_S) for the family $C_{\theta,p}^{A_1,A_2}$ is given by

$$\rho_S = 12\sum_{k=1}^{g}\binom{p}{k}\theta^k B_1(k)B_2(k), \qquad \textbf{(3.2)}$$

where $B_i(k) = \int_0^1 x\, A_i^{\ k}(x)dx$, for $i = 1,2$.

Proof: By using *Eq. (3.1)*, the Spearman's ρ_S can be expanded as

$$\begin{aligned}
\rho_S &= 12\int_0^1\int_0^1 C_{\theta,p}^{A_1,A_2}(u,v)dudv - 3 \\
&= 12\int_0^1\int_0^1 uv[1+\theta A_1(u)A_2(v)]^p dudv - 3 \\
&= 12\int_0^1\int_0^1 uv\sum_{k=0}^{g}\binom{p}{k}\theta^k A_1^{\ k}(u)A_2^{\ k}(v)dudv - 3 \\
&= 12\int_0^1\int_0^1 \left\{uv + \sum_{k=1}^{g}\binom{p}{k}\theta^k u\, A_1^{\ k}(u)\, v\, A_2^{\ k}(v)\right\}dudv - 3 \\
&= 12\left\{\int_0^1\int_0^1 uvdudv + \int_0^1\int_0^1\sum_{k=1}^{g}\binom{p}{k}\theta^k uA_1^{\ k}(u)vA_2^{\ k}(v)dudv\right\} - 3 \\
&= 12\left\{\sum_{k=1}^{g}\binom{p}{k}\theta^k\int_0^1\int_0^1 uA_1^{\ k}(u)vA_2^{\ k}(v)dudv\right\} \\
&= 12\left\{\sum_{k=1}^{g}\binom{p}{k}\theta^k\left(\int_0^1 uA_1^{\ k}(u)du\right)\left(\int_0^1 vA_2^{\ k}(v)dv\right)\right\} \\
&= 12\sum_{k=1}^{g}\binom{p}{k}\theta^k B_1(k)B_2(k). \ \blacksquare
\end{aligned}$$

3.2 | Kendall's Tau

In terms of the copula, Kendall's tau τ_k is defined as (see Nelsen [3]).

$$\tau_k = 4\int_0^1\int_0^1 c(u,v)C(u,v)dudv - 1.$$

Proposition 3. Let (X,Y) be a pair of random variables with the family $C_{\theta,p}^{A_1,A_2}$ and the family density $c_{\theta,p}^{A_1,A_2}$. Kendall's tau (τ_k) can be expanded as

$$\tau_k = 4\sum_{k=0}^{g}\binom{2p-2}{k}\theta^k \left\{ B_1(k)B_2(k) + 2\theta\left(1-\frac{2p}{(k+1)^2}\right)B_1(k+1)B_2(k+1) \right.$$
$$\left. +\theta^2\left(\frac{k+2-2p}{k+2}\right)^2 B_1(k+2)B_2(k+2) \right\} - 1 . \qquad \textbf{(3.3)}$$

Proof: For the sake of textual unity, the proposition's proof is deferred to the Appendix. ■

Remark 2. For $A_i(x) = 1-x$, $i=1,2$, $\forall x \in [0,1]$, and $p=1$ in *Eq. (2.2)*, we have for the classical FGM copula as discussed in Farlie [4], Gumbel [5], and Morgenstern [6] that $\rho_S = \theta/3$ and $\tau_k = 2\theta/9$. Hence, we have $-0.33 \le \rho_S \le 0.33$ and $-0.22 \le \tau_k \le 0.22$ (as $-1 \le \theta \le 1$).

As *Remark 2* shows, the domain of correlation of FGM copula is limited, and therefore it is not allowed for the modeling of strong dependence. One of the advantages of the family $C_{\theta,p}^{A_1,A_2}$ is the capability to improve the domain of correlation by introducing an additional parameter p in FGM copula, and some generalized FGM families presented in recent years.

Example 1. In the family $C_{\theta,p}^{A_1,A_2}$, let $A_i(x) = 1-x$, for $i=1,2$, and $\forall x \in [0,1]$. Then the family $C_{\theta,p}^{A_1,A_2}$ leads to a new symmetric generalized FGM copula with $-(\max\{1,p\})^{-1} \le \theta \le p^{-1}$. Since $B_i(k) = \int_0^1 x A_i^{\,k}(x)dx = \frac{1}{(k+1)(k+2)}$, we have by using *Eq. (3.2)* that

$$\rho_S = 12\sum_{k=1}^{g}\binom{p}{k}\theta^k\left[\frac{1}{(k+1)(k+2)}\right]^2 .$$

where the upper bound of the above Spearman's ρ_S can be increased up to approximately 0.3805 as $p \to \infty$, while the lower bound -0.3333 remains unchanged. Therefore, the admissible range of Spearman's ρ_S in the new symmetric generalized FGM family is $[-0.3333, 0.3805]$.

Example 2. In the family $C_{\theta,p}^{A_1,A_2}$, let $A_i(x) = 1-x^\alpha$, for $i=1,2$, $\forall x \in [0,1]$, and $\alpha \succ 0$. Then the family $C_{\theta,p}^{A_1,A_2}$ leads to a new symmetric generalized Hung-Kotz family with $-\left(\max\{1, p\alpha^2\}\right)^{-1} \le \theta \le (p\alpha)^{-1}$.

Since $B_i(k) = \int_0^1 x A_i^{\,k}(x)dx = \frac{\Gamma(k+1)\Gamma(2/\alpha)}{\alpha\Gamma(k+1+2/\alpha)}$, for $i=1,2$, we have

$$\rho_S = 12\sum_{k=1}^{g}\binom{p}{k}\theta^k\left(\frac{\Gamma(k+1)\Gamma(2/\alpha)}{\alpha\Gamma(k+1+2/\alpha)}\right)^2 . \qquad \textbf{(3.4)}$$

Taking $\alpha \cong 1.85$ and $p > 500$ in *Eq. (3.4)*, we have $\rho_{S,\max} = 0.43$. Similarly, taking $\alpha \cong 0.1$ and $p > 500$, we obtain $\rho_{S,\min} \cong -0.50$. Therefore, the admissible range of Spearman's ρ_S in the generalized Hung-Kotz family is $[-0.50, 0.43]$. So, the generalized Hung-Kotz copula improves the amplitude ρ_S of the Hung-Kotz family.

Remark 3. For $p=1$, *Eq. (3.4)* reduces to $\rho_S = 3\theta\left(\frac{\alpha}{\alpha+2}\right)^2$ whose range is

$$-3\left(\max\{1,\alpha^2\}\right)^{-1}\left(\frac{\alpha}{\alpha+2}\right)^2 \le \rho_S \le 3\frac{\alpha}{(\alpha+2)^2},$$

which is the same as the one discussed by Huang and Kotz [7].

4 | Concepts of Dependence

In this section, we focus on some concepts of dependence for the generalized FGM copula. Several concepts of positive (negative) dependence and dependence stochastic orders have been introduced in the literature (Nelsen [3]). In the following definition, we recall some of these concepts and then study the dependence structure of the family $C_{\theta,p}^{A_1,A_2}$ given in *Eq. (2.2)*.

Definition 2. The random variables X and Y are

I. Positively Quadrant Dependent (PQD) if

$$P(X \le x, Y \le y) \ge P(X \le x)P(Y \le y) \ ,$$

for all $(x,y) \in \Re^2$ or equivalently

$$C(u,v) \ge uv\,,\ \forall (u,v) \in [0,1]^2. \tag{4.1}$$

II. Left Tail Decreasing (LTD) if $P(Y \le y | X \le x)$ is non-increasing in x for all y, or equivalently,

$$u \to \frac{C(u,v)}{u}, \tag{4.2}$$

is non-increasing for all $v \in I$.

III. Left Corner Set Decreasing $(LCSD)$ if $P(X \le x, Y \le y | X \le x_1, Y \le y_1)$ is non-increasing in x_1 and y_1 for all x and y, or equivalently, C is a totally positive function of order $2(TP_2)$, for all $(u_1, u_2, v_1, v_2) \in I^4$ with $u_1 \le u_2$ and $v_1 \le v_2$ if

$$\Lambda = C(u_1,v_1)C(u_2,v_2) - C(u_1,v_2)C(u_2,v_1) \ge 0. \tag{4.3}$$

Proposition 4. Let (X,Y) be a pair of random variables with the family $C_{\theta,p}^{A_1,A_2}$. The random variables X and Y are.

I. PQD if and only if $\theta \ge 0$.

II. LTD if and only if $\theta A_1'(u) \ge 0$.

III. $LCSD$ if and only if both A_1 and A_2 be decreasing or increasing.

Proof:

I. Using *Eq. (4.1)*, the proof is straightforward.

II. Based on *Eq. (4.1)*, is non-increasing with respect to if and only if,

$$\frac{\partial}{\partial u} v\left[1+\theta A_1(u)A_2(v)\right]^p = p\theta A_1'(u) v A_2(v)\left[1+\theta A_1(u)A_2(v)\right]^{p-1} \le 0,$$

on a necessary and sufficient condition that $\theta A_1'(u) \ge 0$.

III. Suppose $u_1 \le u_2$ and $v_1 \le v_2$. By *Eq. (4.3)*, we have

$$\Lambda = u_1 v_1 u_2 v_2 \Big\{ \left(\left[1+\theta A_1(u_1)A_2(v_1)\right]\left[1+\theta A_1(u_2)A_2(v_2)\right]\right)^p$$
$$-\left(\left[1+\theta A_1(u_2)A_2(v_1)\right]\left[1+\theta A_1(u_1)A_2(v_2)\right]\right)^p \Big\} \ge 0.$$

By simple computations, we have $\Lambda \ge 0$ if and only if

$$\left(A_1(u_1)-A_1(u_2)\right)\left(A_2(v_1)-A_2(v_2)\right)\geq 0. \tag{4.4}$$

The relation *Eq. (4.4)* holds if and only if both A_1 and A_2 be decreasing or increasing. ■

As an example, for $A_i(x)=1-x$, $i=1,2$ and $\forall x\in[0,1]$, we have that the classical FGM copulas are PQD, if $0\leq\theta\leq 1$, LTD if $-1\leq\theta\leq 0$ and LCSD for all $-1\leq\theta\leq 1$. Recently, Bairamov and Bayramoglu [16] showed that if in the Baker's model, one uses the dependent random variables (X,Y) with Positive Quadrant Dependent (PQD) joint distribution function $F(x,y)$, instead of independent random variables, then the correlation increases, and in contrast, for Negative Quadrant Dependent (NQD) $F(x,y)$, it decreases.

Lai and Xie [9] stated conditions for having PQD and studied a class of bivariate uniform distributions having PQD property by generalizing the uniform representation of a well-known FGM copula. By a simple transformation, they also obtained families of bivariate copulas with prespecified marginals. We showed that the association parameter in this class can improve the PQD property for a wider range. Thus, we believe that the family $C_{\theta,p}^{A_1,A_2}$ is more applicable in a greater variety of situations.

5 | Conclusion

This paper introduced a generalization of the asymmetric FGM copula family, along with several new extensions. General formulas for association measures, including Kendall's tau and Spearman's rho, were systematically derived for these copulas. Furthermore, necessary and sufficient conditions for key dependence concepts, such as positive quadrant dependence, were rigorously established within this generalized framework. A fundamental limitation of the standard FGM copula is its restricted correlation range, which constrains its practical applicability. The proposed generalized FGM family successfully overcomes this limitation by significantly improving the correlation coefficient's range. Consequently, the primary advantage of this new family lies in its enhanced capability to model a broader and more flexible spectrum of dependence structures. This advancement allows for the extension of the FGM family's potential applications across various scientific disciplines, including finance, hydrology, biostatistics, and reliability engineering, where capturing complex dependence patterns is crucial.

Authors' Contributions

All aspects of the research and manuscript preparation were carried out by the author. The author has read and approved the final version of the manuscript.

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All data supporting the reported findings in this research paper are provided within the manuscript.

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