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A Robust Directional Distance Function for Fixed-Sum Undesirable Outputs: A Petrochemical Industry Study

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Abstract

Petrochemical industries, as energy-intensive and pollution-oriented sectors, operate under stringent environmental regulations, particularly regarding caps on total pollutant emissions. Under such conditions, the fixed-sum property of undesirable outputs creates interdependence among Decision-Making Units (DMUs) and violates the independence assumption of classical Data Envelopment Analysis (DEA) models. Moreover, operational data in these industries are inherently subject to uncertainty due to fluctuations in operating conditions and environmental variability, which may lead to instability in the estimated efficiency frontier. This study proposes a robust equilibrium framework based on the Directional Distance Function (DDF) to evaluate the environmental-economic performance of petrochemical complexes. In the proposed model, the fixed-sum constraint on undesirable outputs is explicitly incorporated into the production technology, and data uncertainty is addressed through a robust optimization approach. The model was applied to data from 36 active petrochemical complexes in 2024. The results indicate that the number of efficient units decreases from 7 in the classical DEA model to 1 in the robust equilibrium model, demonstrating increased discrimination power and improved realism of the proposed framework. Sensitivity analysis with respect to different uncertainty levels confirms the stability of the ranking results. The findings suggest that neglecting the fixed-sum property of undesirable outputs and data uncertainty may lead to misleading and unstable efficiency identification. The proposed framework can serve as an effective decision-support tool for managerial planning and environmental policy-making in the petrochemical industry.

Keywords: Data envelopment analysis, Equilibrium efficiency frontier, Petrochemical industry, Fixed-sum undesirable outputs, Directional distance function, Robust optimization.

1 | Introduction

In the domain of industrial engineering, the assessment and enhancement of performance within complex manufacturing systems, particularly under conditions characterized by resource limitations, conflicting

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objectives, and systemic constraints, constitutes a fundamental challenge for managerial decision-making. Real-world industrial systems, especially those that are large-scale and energy-intensive, typically comprise a collection of manufacturing units. Each unit operates with multiple inputs and outputs, and its performance is shaped not only by its own internal decisions but also by overarching policies and constraints imposed at the industry or regional level. In such contexts, the application of analytical tools grounded in mathematical programming becomes critically important for measuring efficiency, comparing the performance of units, and supporting strategic decisions.

The petrochemical sector, as a strategic and highly energy-intensive industry, occupies a pivotal position in the supply chains of energy and basic materials. These industries generate a diverse array of high-value-added products through the consumption of substantial hydrocarbon feedstocks, energy, and human resources. Concurrently, they produce significant volumes of undesirable environmental outputs, including atmospheric pollutants, industrial effluents, and greenhouse gases. From the standpoint of industrial engineering, evaluating the performance of such systems based exclusively on economic or production metrics, such as output volume or cost, fails to deliver a comprehensive and accurate representation of their true efficiency. Consequently, it is imperative to integrate both economic and environmental dimensions into the performance evaluation process in a holistic and unified manner [1], [2].

Within this paradigm, Data Envelopment Analysis (DEA) has emerged as one of the most prominent methodologies based on linear programming. DEA facilitates the assessment of the relative efficiency of Decision-Making Units (DMUs) that utilize multiple inputs to produce multiple outputs, without necessitating an explicit specification of the underlying production function. In its classical formulation, DEA assumes that each. DMU_j utilizes an input vector $X_j \in R^m$ for the product vector of desirable outputs $Y_j \in R^s$ and the Production Possibility Set (PPS) is defined as follows:

$$\{(X, Y)|X \text{ product } Y\} \tag{1}$$

Owing to this structural adaptability, DEA has been extensively applied to assess the performance of energy-intensive industries, including petrochemicals, steel manufacturing, cement production, and power generation. Nevertheless, a major shortcoming of conventional DEA models lies in their failure to account for undesirable outputs and their underlying assumption of independence among DMUs. This assumption is frequently violated in practical industrial contexts, particularly those governed by macro-level environmental regulations [3], [4].

In the petrochemical industry specifically, the production process inevitably generates not only desirable outputs but also a range of undesirable environmental by-products. If the vector of undesirable outputs for a given DMU_j as $b_j \in R^q$. It is shown that the PPS is defined in the following expanded form.

$$\{(X, Y, b)|X \text{ product } b \text{ and } Y\} \tag{2}$$

In numerous countries, environmental regulations are structured such that the total allowable emission of pollutants at the industry or industrial zone level is established as a fixed aggregate. This binding limit for a specific pollutant k can be formally expressed as follows:

$$\sum_{j=1}^{n} b_{kj} = \bar{B}_k \forall k \tag{3}$$

$\bar{B}_k$ denotes the predetermined total allowable emission for the industry. The presence of such a constant-sum constraint implies that any increase in pollution levels by one petrochemical complex must be offset by a corresponding reduction in the emissions of other complexes within the same system. From an industrial engineering perspective, this characteristic induces interdependence among DMUs, thereby transforming the system from a collection of independent entities into an integrated network. Consequently, the application of conventional DEA models predicated on independent frontiers may yield unrealistic and potentially misleading outcomes, as the performance improvement of a single unit is assessed without accounting for its consequential impact on other units or the system as a whole [1].

To overcome this limitation, the concept of an equilibrium efficiency frontier has been introduced within the DEA literature. In this framework, rather than evaluating each unit against its own distinct efficiency frontier, a common frontier is established for the entire system. This frontier explicitly incorporates the interdependencies among units arising from the constant-sum constraint. The equilibrium efficiency frontier characterizes a state in which no individual unit can enhance its performance unless such an improvement is accompanied by a decrement in the performance of other units, all within the boundaries imposed by the macro-level system constraint. This concept represents a significant advancement in aligning DEA methodologies more closely with the practical realities of complex industrial systems.

Notwithstanding the progress achieved in the domain of equilibrium efficiency frontiers, the majority of existing models treat data as deterministic and radially conceptualize performance improvements. In the context of the petrochemical industry, performance enhancements typically occur in a non-proportional fashion across feed consumption, increased product output, and emission reduction. For instance, substantial emission reductions may be realized with only minor adjustments in production levels, or conversely, production increases might be achieved without commensurate changes in the consumption of certain inputs. This characteristic underscores the necessity for analytical tools capable of modeling non-radial improvements [5], [6].

In response to this requirement, the Directional Distance Function (DDF) has been introduced as a robust instrument in efficiency analysis. This function facilitates the concurrent expansion of desirable outputs and contraction of undesirable outputs along a predefined improvement path. Formally, the DDF is generally defined as follows [5], [7]:

$$\vec{D}(x, y, b; g) = \max_{\beta} \quad \{\beta \mid (x - \beta g_x,\ y + \beta g_y,\ b - \beta g_b) \in T\}, \tag{4}$$

where x, y, b denote vectors of inputs, desirable outputs, and undesirable outputs, respectively; $g = (g_x, g_y, g_b)$ represents the direction vector specifying the scaling of inputs and outputs; β is the efficiency score indicating the maximal feasible expansion and contraction along the given direction; and T denotes the PPS.

Nevertheless, the application of the DDF within the framework of the equilibrium efficiency frontier and in the presence of fixed-sum undesirable outputs has thus far received limited scholarly attention. Integrating these two concepts necessitates the development of a model that simultaneously accounts for the interdependence among DMUs and accommodates non-radial performance improvements, a task that presents considerable challenges in both mathematical formulation and managerial interpretation [8].

Beyond these structural considerations, the data employed in assessing the performance of petrochemical industries are typically characterized by uncertainty. Variations in operational conditions, fluctuations in feedstock and energy prices, unplanned unit shutdowns, and measurement inaccuracies all contribute to deviations between actual and nominal data values. Classical DEA models, which assume deterministic data, exhibit substantial sensitivity to such fluctuations and may consequently generate unstable efficiency frontiers and unreliable evaluative outcomes. This issue underscores the critical importance of adopting robust optimization approaches in efficiency assessment [9].

Accordingly, a discernible gap exists in the literature regarding a comprehensive analytical framework capable of concurrently addressing: 1) the interdependence among units arising from the constant-sum constraint on undesirable outputs, 2) the non-radial nature of performance improvements, and 3) the uncertainty inherent in operational data within the context of petrochemical industry efficiency evaluation.

The present study, grounded in an industrial engineering perspective, aims to introduce an innovative DEA model that integrates the equilibrium efficiency frontier, the DDF, and robust optimization methodologies.

This integrated framework is intended to provide a robust analytical tool for examining and enhancing the performance of complex production systems, thereby offering a scientific foundation for managerial decision-making and environmental policy formulation in the petrochemical sector.

2 | Methodology

This study is applied in terms of its objective and analytical-developmental in terms of its nature. In this research, a robust equilibrium model based on the DDF was developed within the framework of DEA and subsequently implemented using real data from active petrochemical complexes in Iran. The research process was conducted systematically through several sequential stages, including data collection, variable selection, definition of the equilibrium structure, model formulation, numerical solution, and ultimately, result analysis.

2.1 | Statistical Population and Data Collection

The statistical population of this research comprises active petrochemical complexes operating in Iran during the year (2024-2025). Following an examination of the operational status of the units and verification of data completeness, 36 active complexes possessing accurate and reliable information were selected as DMUs. The dataset encompassed feedstock consumption, energy consumption, primary production rate, and CO_2 emissions. Data were extracted from annual performance reports, industry databases, and official statistical records. To ensure the applicability of the DEA methodology, descriptive statistics of the variables, including mean, standard deviation, and range of variation- were calculated to assess the relative heterogeneity among units and the discriminatory power of the data [9].

2.2 | Variable Selection and Model Structure

In the subsequent phase, model variables were selected in accordance with the inherent characteristics of production processes in the petrochemical industry and based on the extant literature on efficiency analysis of energy-intensive industries. Within this framework, feedstock consumption (measured in tons) and energy consumption (measured in MWh) were designated as the primary inputs. The production rate (measured in tons) was considered the desirable output, while CO_2 emissions (measured in tons) were incorporated as an undesirable output. This variable selection was deliberately structured to simultaneously capture both the economic and environmental dimensions of performance, thereby reflecting the actual technological structure of petrochemical production [6].

Owing to environmental regulations imposed at the industry level, the undesirable output, CO_2 emissions, was characterized by a constant-sum property at the system level. Accordingly, the constant-sum emission constraint was explicitly integrated into the structure of the production technology. Rather than defining independent efficiency frontiers for each unit, a common equilibrium efficiency frontier was established for the entire system. This equilibrium frontier explicitly accounts for the interdependence among complexes within the framework of the aggregate emission ceiling, thereby elevating the performance assessment from the unit level to the systemic level [1].

2.3 | Directional Distance Function and Efficiency Measurement

To measure the distance of each complex from the equilibrium efficiency frontier, the DDF was employed. This approach facilitates the simultaneous expansion of desirable outputs and contraction of undesirable outputs along a predefined improvement path. At this stage, the equilibrium directional inefficiency index (β) and the equilibrium efficiency index (θ) were computed for each DMU [5], [7].

2.4 | Robust Optimization Framework

Given the inherent uncertainty associated with industrial data and the presence of operational fluctuations, input and output data were modeled as bounded intervals around their nominal values. To accommodate this uncertainty, the Bertsimas and Sim [10] robust optimization framework was adopted. Within this framework, the equilibrium model constraints were reformulated to remain valid even under the worst-case data deviations. The uncertainty budget was controlled through the parameter Γ, and the impact of varying uncertainty levels on the evaluation outcomes was systematically examined. The output of this stage was the robust equilibrium efficiency index (θ^R), which indicates the distance of each unit from the system efficiency frontier under both the constant-sum constraint and data uncertainty.

2.5 | Model Implementation and Solution

Subsequently, the robust equilibrium model was solved individually for each of the 36 petrochemical complexes. The model solution process was executed using mathematical programming software, specifically GAMS with the CPLEX solver. Following the solution procedure for all units, the values β^R and θ^R were extracted, and the ranking of complexes was determined based on the θ^R index [9].

2.6 | Analytical Framework

In the final stage, the results obtained from model implementation were analyzed from three distinct perspectives: first, analysis of the ranking of complexes based on the robust equilibrium efficiency index; second, comparison of the results with those derived from classical DEA models, with particular attention to discrepancies in the identification of efficient units; and third, sensitivity analysis with respect to different levels of the uncertainty parameter Γ to examine the stability of rankings under increasing uncertainty conditions. These analyses facilitated the provision of managerial and policy-oriented interpretations regarding the performance status of the petrochemical industry and enabled an applied evaluation of the proposed analytical framework [11].

3 | Theoretical Foundations

DEA, as one of the most important achievements of mathematical programming in the field of performance evaluation, provides a nonparametric framework for estimating the empirical production frontier in which the relative efficiency of DMUs can be estimated without the need to specify. The data structure is defined as follows:

$$X_j = (x_{1j}, x_{2j}, \dots, x_{mj}) \in \mathbb{R}_+^m$$
$$Y_j = (y_{1j}, y_{2j}, \dots, y_{sj}) \in \mathbb{R}_+^s$$

The PPS is defined as a convex hull of observations:

$$T_v = \left\{(X, Y) \middle| \sum_{j=1}^{n} \lambda_j X_j \le X, \sum_{j=1}^{n} \lambda_j Y_j \ge Y, \sum_{j=1}^{n} \lambda_j = 1, \lambda_j \ge 0, j = 1, \dots, n \right\} \tag{5}$$

The BCC model is proposed as follows.

$$\begin{aligned} &\min \theta \\ &\text{s.t.} \\ &\sum_{j=1}^{n} \lambda_j X_j \le \theta X_o \\ &\sum_{j=1}^{n} \lambda_j Y_j \ge Y_o \\ &\sum_{j=1}^{n} \lambda_j = 1 \\ &\lambda_j \ge 0, j = 1, \dots, n \end{aligned} \tag{6}$$

The optimal value of the technical efficiency index, denoted as $0 < \theta^* \leq 1$, indicates the performance status of a DMU. If $\theta^* = 1$, the unit is classified as efficient; otherwise, it is deemed inefficient [11].

In energy-intensive industries such as petrochemicals, steel manufacturing, cement production, and power generation, the following characteristic features are commonly observed:

1) Multiple energy-intensive and feedstock-based inputs, 2) Simultaneous production of multiple outputs, 3) Complexity of cost structures, 4) Dependence on macroeconomic energy and environmental policies.

Under such conditions, the principal advantage of DEA resides in its capacity to compare relative performance without necessitating the parametric estimation of a production function. Numerous empirical studies conducted across Chinese, European, and American industries have demonstrated that DEA constitutes an effective tool for analyzing energy efficiency and environmental performance [1].

Nevertheless, the classical CCR and BCC models are predicated upon assumptions that are not entirely valid in complex industrial environments [12]. These assumptions include: Independence of DMUs; Absence of macro-level system constraints; Deterministic data; Unrestricted freedom to modify output levels.

These underlying assumptions are substantially violated in petrochemical industries operating under binding emission caps and regulatory policies, thereby limiting the applicability of conventional DEA models in such contexts [13].

In numerous industrial production systems, particularly within energy-intensive sectors such as petrochemicals, the production process inevitably generates undesirable environmental outputs. These by-products encompass a range of atmospheric pollutants, industrial effluents, and greenhouse gases that are produced concurrently with the primary desirable outputs. From the perspective of efficiency analysis, the presence of such outputs introduces considerable complexity into the analytical problem. Unlike desirable outputs, which firms typically seek to maximize, the reduction of undesirable outputs generally entails incurring additional costs, deploying abatement technologies, or even accepting reductions in production levels. Consequently, undesirable outputs cannot be modeled analogously to conventional outputs within the classical DEA framework [5]. Suppose that for each $\mathrm{DMU_j}$ The vector of undesirable outputs is defined as follows:

$$b_j = (b_{1j}, b_{2j}, \dots, b_{qj}) \in \mathbb{R}^q_+,$$

where b_{kj} denotes the quantity of undesirable output k generated by $\mathrm{DMU_j}$, and q represents the number of undesirable output categories under consideration. When incorporating undesirable outputs into the analytical framework, the production technology set is defined in the following expanded formulation [5]:

$$T_v = \left\{(X, Y, b) \middle| \sum_{j=1}^{n} \lambda_j X_j \leq X, \sum_{j=1}^{n} \lambda_j b_j \leq b,\ \sum_{j=1}^{n} \lambda_j Y_j \geq Y,\ \sum_{j=1}^{n} \lambda_j = 1, \lambda_j \geq 0, j = 1, \dots, n \right\} \tag{7}$$

In *Eq. (1)*, the condition $\sum_{j=1}^{n} \lambda_j b_j \leq b$ embodies the fundamental principle that undesirable outputs are not freely disposable. More precisely, this constraint formally expresses the notion that the reduction of pollutants is not achievable without incurring corresponding costs or without necessitating a concomitant reduction in production activity. This characteristic, known as weak disposability of undesirable outputs, constitutes the essential theoretical distinction between environmental DEA models and their classical counterparts [5], [6].

3.1 | Existing Approaches to Modeling Undesirable Outputs

Within the DEA literature, three principal approaches have been proposed for addressing undesirable outputs in efficiency evaluation [6].

3.1.1 | Treating undesirable outputs as inputs

In this approach, pollutants are introduced into the model as inputs rather than outputs. Although this method is computationally straightforward, it lacks both economic and technical justification, as pollutants are inherently the by-products of the production process rather than causative factors. Consequently, this transformation may distort the interpretation of efficiency measures and alter the actual structure of the underlying production technology [6].

3.1.2 | Mathematical transformation of undesirable outputs

In this method, undesirable outputs are converted into desirable outputs through mathematical transformations, such as the reciprocal transformation:

$$b' = \frac{1}{b} \tag{8}$$

While such transformations enable the application of the classical DEA framework, they fundamentally distort the true relationship between pollutant generation and emissions. Furthermore, these transformations modify the characteristics of returns to scale, thereby artificially altering the technological structure and potentially leading to misleading efficiency assessments [14].

3.1.3 | Weak disposability assumption

The third approach, which possesses a more robust theoretical foundation, is predicated upon the concept of weak disposability. Within this framework, the production technology satisfies the following condition [5]:

If

$(x, y, b) \in T$ and $0 \leq \alpha \leq 1,$

Then

$(x, \alpha y, \alpha b) \in T$

This condition formally expresses that reducing undesirable outputs is necessarily accompanied by a proportional reduction in desirable outputs, reflecting the technological reality that pollution abatement entails opportunity costs in terms of foregone production. The weak disposability assumption has gained widespread acceptance in environmental efficiency analysis due to its consistency with the physical laws governing production processes and its ability to preserve the technological relationships embedded in the data [6].

Eqs. (2)-(10) formally express that the reduction of undesirable outputs is feasible only if the level of desirable outputs is simultaneously reduced in the same proportion. This condition implies that production and pollution share a common and inherently interrelated nature; complete elimination of pollutants is not technologically possible without fundamentally altering the production process itself [5]. This assumption aligns closely with the technical reality of the petrochemical industry, where chemical reactions inevitably generate both desired products and unwanted by-products as integral outcomes of the same transformation process [6].

3.2 | Limitations of Slack-Based Models

As the literature has evolved, Slack-Based Measures (SBM) were introduced to provide more accurate assessments of inefficiency in the presence of undesirable outputs. These models enable direct measurement of excess emissions and are characterized by a non-radial structure that captures inefficiencies in individual inputs and outputs separately [15]. However, the majority of these models operate under the assumption that the aggregate quantity of undesirable outputs is freely variable at the system level.

In industrial contexts subject to stringent environmental regulations, this assumption fails to hold. In numerous countries, the total emission of pollutants at the industry or industrial zone level is established as a fixed ceiling or binding constraint. Under such circumstances, undesirable outputs are constrained not only at the individual DMU level but also at the systemic level, and any reduction in emissions by one unit must necessarily be accompanied by reallocation or compensatory adjustments among other units [1].

Consequently, the mere application of weak disposability assumptions or SBM models is insufficient for analyzing such systems. It becomes imperative to explicitly incorporate the fixed-sum constraint into the technological structure of the production model. This requirement motivates the development of the equilibrium efficiency frontier concept, which is elaborated in the subsequent section as a framework for addressing the interdependencies arising from system-level emission constraints.

3.3 | Undesirable Outputs with Constant Sum and the Equilibrium Efficiency Frontier

In numerous industries operating under environmental regulatory frameworks, the total emission of pollutants at the industry or industrial zone level is determined and monitored by regulatory authorities. This characteristic implies that undesirable outputs are constrained not only at the individual DMUs level but also at the systemic level. Such a situation fundamentally diverges from the classical DEA framework, wherein each unit is optimized independently without consideration of interdependencies among units [16], [17]. Suppose that for pollutant k, the total system emission ceiling is denoted by $\bar{B}_k$. This system-level constraint can be formally expressed as:

$$\sum_{j=1}^{n} b_{kj} = \bar{B}_k \forall k$$

This constraint constitutes a coupling system constraint, as the emission decision of each unit directly influences the feasible decision space of all other units within the system. Under such conditions, improving the environmental performance of a particular DMU necessitates the reallocation of emission entitlements throughout the entire system, thereby introducing interdependencies that conventional DEA models fail to capture [18].

3.4 | Zero-Sum Gains Data Envelopment Analysis Model

The first formal analytical framework developed to address fixed-sum outputs was the Zero-Sum Gains (ZSG) DEA model. The fundamental premise underlying this approach is that if a given unit reduces its emissions, the quantity of emission reduction must be redistributed proportionally among the other units in the system to maintain the aggregate emission constraint. If Δb_o represents the emission reduction achieved by DMU_o, the reallocation of emissions to other units is defined according to a proportional adjustment mechanism. This reallocation ensures that the total system emission level remains constant while individual units adjust their emission levels in accordance with their relative contributions to the overall system [16].

The ZSG-DEA framework thus provides a foundation for modeling efficiency in contexts where undesirable outputs are subject to fixed-sum constraints, recognizing that environmental improvements by one unit must be balanced by corresponding adjustments elsewhere in the system.

$$b_j^{new} = b_j + \frac{b_j}{\sum_{i \neq o} b_i} \Delta b_o \tag{9}$$

3.5 | Fixed-Sum Output Data Envelopment Analysis Model

Subsequently, fixed-sum output models were developed in which the system-level emission constraint, represented by *Eqs. (2)-(10)*, is directly incorporated into the linear programming formulation. Within this analytical framework, the production technology is redefined as follows [19]:

$$T^{FS} = \left\{ (x, y, b) \mid x \geq X\lambda, y \leq Y\lambda, b \geq B\lambda, \sum_{j=1}^{n} b_{kj} = \bar{B}_k, \lambda \geq 0 \right\} \tag{10}$$

Despite explicitly incorporating the fixed-sum constraint, these models continue to evaluate efficiency relative to local boundaries, and complete system-level equilibrium is not necessarily achieved. The individual frontiers remain conceptually distinct, and the interdependencies introduced by the coupling constraint are not fully resolved in terms of establishing a system-wide equilibrium [6].

3.5 | Equilibrium Efficiency Frontier

To address this limitation, the concept of an equilibrium efficiency frontier was introduced in the DEA literature. In this framework, rather than evaluating each DMU against its own independent frontier, a common equilibrium frontier is established at which the entire system resides in an optimal equilibrium state. This approach explicitly recognizes that under fixed-sum constraints, the efficiency of any individual unit cannot be meaningfully assessed in isolation from the system as a whole [1].

In the Generalized Equilibrium Efficiency Frontier Data Envelopment Analysis (GEEFDEA) model developed by Yang et al. [1], the overall structure is formulated to simultaneously satisfy both the fixed-sum constraint and the condition that no unit can improve its performance without adversely affecting the performance of other units within the system. The equilibrium frontier thus represents a Pareto-optimal state for the entire collection of DMUs subject to the system-level emission ceiling.

$$\begin{aligned}
&\min \theta \\
&\text{s.t.} \textstyle\sum_{j=1}^{n} \lambda_j X_j \leq \theta X_o \\
&\textstyle\sum_{j=1}^{n} \lambda_j Y_j \geq Y_o \\
&\textstyle\sum_{j=1}^{n} \lambda_j b_j = \bar{B}_k \\
&\textstyle\sum_{j=1}^{n} \lambda_j = 1 \\
&\lambda_j \geq 0, j = 1, \dots, n
\end{aligned} \tag{11}$$

Constraint $\sum_{j=1}^{n} \lambda_j b_j = \bar{B}_k$ ensures that the total emission level of the entire system remains constant. The defining characteristics of this framework are as follows:

The efficiencies of individual units are no longer independent of one another. The efficiency frontier is determined as a function of the aggregate emission distribution across the system. Performance improvement by any single unit necessitates optimal reallocation of emission entitlements throughout the entire system [1].

3.6 | Structural Analysis of the Equilibrium Frontier

Within the equilibrium framework, the evaluation problem can be conceptualized as a two-level optimization problem:

I. Level 1: determine the optimal allocation of emissions across the system;

II. Level 2: evaluate the efficiency of each unit under the newly established allocation.

This hierarchical structure reveals that the equilibrium frontier is, in essence, the outcome of a collective optimization process rather than the aggregation of individually determined frontiers [6].

Geometrically, whereas classical DEA generates an independent convex piecewise linear frontier for each unit, the equilibrium frontier constitutes a common hyperplane constrained by the constant-sum condition. This system-level constraint effectively compresses the PPS, rendering the feasible region more compact and reflecting the technological interdependencies induced by environmental regulations [16].

3.7 | Limitations of Existing Equilibrium Models

Notwithstanding the significant theoretical advances represented by equilibrium efficiency frontier models, the existing formulations exhibit several limitations: Reliance on the assumption of deterministic data, predominant focus on radial measures of improvement, inflexibility in specifying improvement pathways, high sensitivity to data fluctuations and measurement errors, and absence of a robust framework suitable for volatile industries such as petrochemicals [9].

Consequently, although models such as GEEFDEA and Equilibrium Efficiency Frontier Data Envelopment Analysis (EEFDEA) constitute fundamental steps toward incorporating system-level constraints into efficiency analysis, they require further development to enhance their applicability in real-world energy-intensive environments characterized by uncertainty and operational variability.

3.8 | Directional Distance Function in the Framework of the Equilibrium Efficiency Frontier

In complex industrial systems, inefficiencies typically exhibit an asymmetric structure. For example, within the petrochemical industry, it may be feasible to substantially reduce CO_2 emissions without any noticeable change in ethylene production levels; Conversely, increasing production may necessitate a disproportionately large increase in energy consumption relative to the output gain. Classical radial models are inherently incapable of capturing such asymmetrical behavioral patterns. To address this analytical gap, the DDF was introduced into the efficiency measurement literature [7], [20].

The general definition of the DDF is formulated as follows:

$$\vec{D}(x, y, b; g) = \max_{\beta} \{\beta \mid (x - \beta g_x, y + \beta g_y, b - \beta g_b) \in T\} \tag{12}$$

In this formulation:

g_x denotes the direction vector for input contraction;

g_y denotes the direction vector for desirable output expansion;

g_b denotes the direction vector for undesirable output contraction.

The key property of the DDF is expressed by the following implication:

Provided that the scalar

$$(x, y, b) \in T \Rightarrow (x - \beta g_x, y + \beta g_y, b - \beta g_b) \in T$$

β is feasible within the PPS. This property enables the DDF to accommodate non-proportional adjustments across different dimensions of the production process, thereby offering greater flexibility in modeling the complex trade-offs inherent in energy-intensive industries [5].

3.9 | Rewriting the Directional Distance Function in the Equilibrium Framework

Now, if we introduce the constant sum of emissions constraint into the model, the equilibrium production technology will be as follows:

$$T^{EE} = \left\{ (X, Y, b) \mid X \geq X\lambda, Y \leq Y\lambda, b \geq B\lambda, \sum_{j=1}^{n} b_{kj} = \bar{B}_k, \lambda \geq 0 \right\} \tag{13}$$

In this case, the equilibrium DDF is defined as:

$$\begin{aligned}
&\max \beta \\
&\text{s.t.} \sum_{j=1}^{n} \lambda_j X_j \leq X_o - \beta g_x \\
&\sum_{j=1}^{n} \lambda_j Y_j \geq Y_o + \beta g_y \\
&\sum_{j=1}^{n} \lambda_j b_j \leq b_o - \beta g_b \\
&\sum_{j=1}^{n} \lambda_j = 1 \\
&\sum_{j=1}^{n} \lambda_j b_j = \bar{B}_k \\
&\lambda_j \geq 0, j = 1, \dots, n
\end{aligned}$$

This formulation demonstrates that the magnitude of potential improvement, denoted by β, is now contingent upon the system-wide allocation of emissions. Consequently, the inefficiency score of each unit depends not only on its own operational data but also on the distribution of emission entitlements across the entire system [1], [21].

3.10 | Data Uncertainty and Robust Data Envelopment Analysis

In the petrochemical industry, operational data are frequently characterized by uncertainty arising from measurement errors, process fluctuations, and varying operating conditions. Suppose that the inputs are modeled as subject to additive uncertainty in the following manner [22]:

$$x_{ij} = \hat{x}_{ij} + \xi_{ij}, \tag{14}$$

Where $\hat{x}_{ij}$ represents the nominal value of input i for DMU_j and ξ_{ij} denotes the random deviation from this nominal value. The uncertainty is assumed to be bounded within a specified interval:

$$| \xi_{ij} | \leq \Delta_{ij} \tag{15}$$

In which Δ_{ij} defines the radius of uncertainty for the corresponding input parameter [10]. Within the framework of robust optimization, model constraints must remain valid even under the worst-case realizations of the uncertain parameters. For example, the input constraint corresponding to the first equation in the *Model (13)* is reformulated in its robust counterpart as follows:

$$\max_{|\xi_{ij}| \leq \Delta_{ij}} X\lambda \leq x_o - \beta g_x \tag{16}$$

This maximization problem over the uncertainty set leads to the following deterministic formulation [10]:

$$\hat{X}\lambda + \Gamma_x \leq x_o - \beta g_x \tag{17}$$

Represents the price of robustness or the uncertainty budget parameter that controls the degree of conservatism in the solution. Analogously, for desirable outputs, the robust counterpart is expressed as:

$$\hat{Y}\lambda - \Gamma_y \geq y_o + \beta g_y \tag{18}$$

And for undesirable outputs, the corresponding robust constraint takes the form:

$$\hat{B}\lambda + \Gamma_b \leq b_o - \beta g_b \tag{19}$$

These robust formulations ensure that the efficiency evaluation remains valid across a range of possible data realizations, thereby providing stability in the resulting efficiency scores and rankings under conditions of uncertainty [9].

3.11 | Integrated Analysis of three Components: System Equilibrium, Non-Radial Improvement, and Robustness

Synthesizing the discussions presented in the preceding sections, the performance evaluation of petrochemical industries now confronts three fundamental components that must be addressed simultaneously: first, the existence of a constant-sum constraint on undesirable outputs, which induces interdependence among DMUs and necessitates the formation of an equilibrium structure at the system level; second, the non-radial nature of performance improvements, which underscores the necessity of employing a DDF to concurrently model the expansion of desirable outputs and the contraction of undesirable outputs; and third, the presence of uncertainty in operational data, which mandates the adoption of a robust optimization framework to obtain stable and reliable results [1], [9], [21].

Under a fixed aggregate emission constraint, production technology must be defined such that any reduction in the undesirable output of one unit is necessarily accompanied by reallocation to other units within the system. This constraint, previously expressed as: $\sum_{j=1}^{n} b_{kj} = \bar{B}_k$ transforms the problem from a collection of independent evaluations into a system-level optimization problem characterized by coupling constraints. In this context, the DDF must be redefined within the framework of this equilibrium technology, such that the magnitude of potential improvement, β, is determined not solely based on the evaluated unit's own data, but also by considering the structural configuration of the entire system [1], [5], [21]. Simultaneously, given the inherent uncertainty in industrial data, model constraints must remain valid even under worst-case realizations of data deviations.

In other words, the production technology constraints and the constraints governing undesirable outputs must be reformulated to hold for all permissible values of deviation within the specified uncertainty set. This reformulation introduces protective components, such as robustness parameters, into the model constraints and transforms the resulting efficiency frontier into one that is resilient to data fluctuations [10]. Consequently, the integration of the three aforementioned components yields an optimization problem with the following structural characteristics: The presence of coupling constraints arising from the fixed-sum condition, which imposes interdependence among units; The dependence of the inefficiency score on the selected direction vector within the DDF, thereby providing flexibility for improvement analysis. The modification of constraints to ensure validity against worst-case data deviations, thereby incorporating robustness into the efficiency assessment. Within this integrated framework, the overall structure of the model can be represented as follows:

$$\max_{\beta,\lambda} \ \beta \tag{20}$$

Subject to the constraints of the modified equilibrium technology in the presence of uncertainty, including the robust counterparts corresponding to *Constraints (17)-(19)*, together with the constant-sum constraint

$\sum_{j=1}^{n} b_{kj} = \bar{B}_k$. In this model, the optimal value serves as an indicator of the distance of the unit from the system's robust equilibrium efficiency frontier, a frontier that simultaneously accounts for both the interdependence among units and the fluctuations in operational data.

From a theoretical perspective, this structure suggests that efficiency assessment in energy-intensive industries operating under environmental regulations possesses a multilayered nature: the first layer pertains to the geometry of the production frontier; the second layer concerns systemic equilibrium in emission allocation; and the third layer addresses the robustness of results to data uncertainty. Neglecting any of these layers may lead to unrealistic efficiency estimates and unsustainable managerial recommendations [6], [13].

4 | Research Gap Analysis

A systematic review of the extant literature reveals that while each of the existing analytical frameworks has addressed certain dimensions of the performance evaluation problem, none has simultaneously incorporated all the aforementioned aspects within an integrated structure. Classical DEA models, despite constituting powerful tools for evaluating multi-input multi-output efficiency, do not explicitly account for undesirable outputs, fail to incorporate systemic constraints, and operate under the assumption of deterministic data [23]. SBM models have advanced the literature by accommodating undesirable outcomes in a non-radial manner; however, they lack an equilibrium mechanism necessary for addressing fixed-sum constraints [15]. Models based on the ZSG approach and those incorporating fixed-sum constraints have taken important steps toward considering constant-sum conditions, yet they typically rely on radial structures and have not been extended to incorporate robustness against data uncertainty [18], [21].

The GEEFDEA and EEFDEA models have successfully formulated the concept of an equilibrium efficiency frontier in the presence of fixed-sum undesirable outputs. Nevertheless, these models assume deterministic data and exhibit limitations regarding the stability of results under operational fluctuations [1], [19], [21]. Conversely, Robust DEA and Robust DDF models have provided frameworks for addressing data uncertainty, yet they do not systematically consider the interdependencies arising from fixed-sum constraints or the concept of an equilibrium frontier [9], [10].

More specifically, the existing body of literature demonstrates that no model to date has simultaneously integrated the following four key features into a coherent analytical framework: 1) determination of an equilibrium efficiency frontier at the system-wide level, 2) modeling of undesirable outputs subject to fixed-sum constraints, 3) employment of the DDF for non-radial inefficiency analysis, and 4) application of robust optimization to address operational data uncertainty. This absence of simultaneous integration constitutes a significant methodological gap in the performance assessment of energy-intensive industries [3], [4], [6], [13].

5 | Innovations of the Present Study

The primary innovation of this research lies in the development of an integrated analytical framework for evaluating the performance of petrochemical industries under systemic environmental constraints and data uncertainty. Whereas previous studies have addressed the modeling of undesirable outputs, equilibrium efficiency frontiers, DDFs, or robust approaches in isolation, the present research simultaneously and coherently integrates these components into a robust equilibrium model based on the DDF. Specifically, the innovations of this study can be summarized in four principal axes:

Integration of Equilibrium Frontier, DDF, and robust optimization: combining the equilibrium efficiency frontier, the DDF, and robust optimization within a single framework, thereby simultaneously accounting for the interdependence among DMUs arising from the fixed-sum constraint on undesirable outputs and the uncertainty inherent in operational data. Explicit modeling of fixed-sum undesirable outputs: formalizing undesirable outputs with constant-sum properties at the level of the entire petrochemical industry and developing a common equilibrium efficiency frontier that elevates performance assessment from the independent unit level to the system equilibrium level. Introduction of a robust equilibrium efficiency index:

proposing the robust equilibrium efficiency index (θ) as a novel metric for measuring the environmental-economic performance of complexes. This index quantifies the distance of each unit from the system efficiency frontier under the dual conditions of environmental constraints and data uncertainty. Comprehensive sensitivity analysis: conducting sensitivity analysis with respect to the uncertainty budget parameter (Γ) and examining the stability of petrochemical complex rankings under varying levels of uncertainty, thereby enabling an assessment of the robustness of the industry's efficiency structure. Furthermore, the application of the proposed model to real data from 36 active Iranian petrochemical complexes operating in the year (2024-2025) substantiates the applied dimension of this research and demonstrates that the presented framework can serve as a practical tool for managerial decision-making and environmental policy formulation within the petrochemical sector.

6 | Summary of Theoretical Foundations

The synthesis of theoretical discussions presented in Section 2 reveals that although the DEA literature has witnessed significant advances in recent decades concerning the modeling of undesirable outputs, fixed-sum constraints, and robust approaches, these developments have largely progressed in isolation within independent frameworks. Classical models remain incapable of analyzing systemic constraints; equilibrium models have not transcended the assumption of data certainty; robust models have failed to incorporate unit interdependence; and models based on the DDF have typically been employed within non-equilibrium frameworks.

Consequently, the research gap can be precisely and explicitly articulated as follows: the necessity of developing a model that can simultaneously capture 1) the interdependence among DMUs resulting from fixed-sum undesirable output constraints, 2) the non-radial nature of performance improvements, and 3) the uncertainty of operational data, all within a robust equilibrium framework grounded in the DDF. Such a model would not only provide conceptual coherence within the DEA literature from a theoretical perspective but also furnish a more accurate and sustainable tool for evaluating and enhancing the performance of petrochemical industries from a practical standpoint.

7 | Conceptual Framework and Proposed Model

This section aims to provide an integrated mathematical framework for evaluating the performance of petrochemical complexes under real operating conditions; conditions in which, on the one hand, undesirable environmental outputs are subject to macroconstraints and emission ceilings and their sum is assumed to be constant at the industry level, on the other hand, performance improvement is of a non-radial nature and requires simultaneous increase of desirable outputs and reduction of undesirable outputs in disparate paths, and finally, input and output data are subject to uncertainty due to operational fluctuations, technological changes and measurement errors. Therefore, the proposed model of this research is developed by combining three components: equilibrium efficiency frontier, “DDF”, and “robust“ optimization to provide a realistic, stable, and reliable picture of the efficiency of petrochemical systems.

In the first step, in order to initially assess the quality of the data and ensure their suitability for implementing the DEA model, descriptive statistics indices were calculated for all input variables, desirable outputs, and undesirable outputs. These indices included the mean, standard deviation, minimum, maximum, and quartiles, and their results are presented in the table below. As is clear from the results of the table, all variables have a significant range of variation, and a significant difference is observed between their minimum and maximum values. This indicates the relative heterogeneity of DMUs and the existence of sufficient diversity in the structure of resource consumption, production level, and pollution levels of the petrochemical complexes under study, which is one of the basic assumptions of the application of the DEA method. To examine the resolution of the data and ensure the model's ability to distinguish between the performance of DMUs, the coefficient of variation index was used 1 was used, which is defined as the ratio of the standard deviation to the mean. The calculation results showed that the coefficient of variation for most of the variables used in the model is around 0.70, which is significantly larger than the conventional threshold of 0.30 reported in the DEA literature.

This finding indicates that the data dispersion is at an appropriate level and the model has sufficient resolution to distinguish between the performance of petrochemical complexes. As a result, the problem of over-identification of efficient units is not expected to occur in the implementation of the model. Accordingly, it can be concluded that the data used are in a suitable state in terms of dispersion, heterogeneity, and resolution, and the necessary conditions are in place to implement the DEA model, especially the robust equilibrium model based on the DDF. Therefore, the results obtained from implementing the model in the later stages of the research will have sufficient methodological validity. *Table 1* proposes descriptive statistics of model variables.

Table 1. Descriptive statistics of model variables.

Maximum	Minimum	Standard Deviation	Average	Variable
6899524	350160	1594304	2259108	Feed (tons)
5307322	269353	1226387	1737774	Energy (MWh)
11278.1	572.4	2606.1	3692.8	Water (thousandm^3)
11808810	599312	2728714	3866552	Labor (person-hours)
6634166	336692	1532987	2172222	Production (tons)
3506819	177976	810337	1148236	Revenue (Billion Rials)
3980500	202015	919792	1303333	CO_2 (tons)
4643.9	235.7	1073.1	1520.6	NOx (tons)
3317.1	168.3	766.5	1086.1	SO_2 (tons)
796.1	40.4	183.96	260.7	PM10 (tons)
2985.4	151.5	689.8	977.5	COD (tons)

7.1 | Decision-Making Units and Notation

Suppose there is a set of$n$ homogeneous petrochemical complexes as DMU_j. Each DMU complex operates using m inputs s desired outputs, and q undesirable outputs. The data notation is as follows.

x_{ij}: Amount of the i-th input for complex for $i = 1, \dots m$,

y_{rj}: Amount of the r-th desirable output for complex for $r = 1, \dots, s$,

b_{kj}: Amount of the k-th undesirable output for the complex for $k = 1, \dots, q$.

Next, the unit being evaluated $DMU_j, j = 1, \dots, n$, is identified by the index.

7.2 | The Constant Sum of Undesirable Outputs Constraint

In the petrochemical industry, environmental regulations usually set a ceiling on the total emission of pollutants at the industry or industrial zone level. Therefore, the undesirable outputs have a constant sum property and are modeled as follows:

$$\sum_{j=1}^{n} b_{kj} = \bar{B}_k, k = 1, \dots, q$$

This constraint expresses the interdependence of units, such that reducing emissions in one complex necessarily requires increasing the reduction capacity in other complexes or reallocating emissions throughout the system.

7.3 | Robust Equilibrium Model based on the Directional Distance Function

Suppose there is a set of petrochemical complexes as DMUs, each unit n operating x_{ij} using m, inputs s desired outputs, and q undesirable outputs. The data related to inputs, desired outputs, and undesirable outputs are represented b_{kj} , y_{rj} where $i = 1, \dots, m$, $r = 1, \dots, s$, $k = 1, \dots, q$ and $j = 1, \dots, n$. Given the existence of a total emission cap at the industry level, undesirable outputs have a constant sum property, and this constraint is applied as the following relationship:

$$\sum_{j=1}^{n} b_{kj} = B_k^{tot}, k = 1, \dots, q \tag{21}$$

Eq. (21) expresses the interdependence of DMUs and allows the determination of the efficiency frontier to be carried out in the form of an equilibrium framework. Accordingly, the equilibrium production set is defined as follows:

$$T^E = \left\{ (x, y, b) \mid \begin{array}{c} x \geq \sum_{j=1}^{n} \lambda_j x_j \\ y \leq \sum_{j=1}^{n} \lambda_j y_j \\ b \geq \sum_{j=1}^{n} \lambda_j b_j \\ \sum_{j=1}^{n} \lambda_j = 1 \\ \sum_{j=1}^{n} b_{kj} = B_k^{tot} \end{array} \right\} \tag{22}$$

To measure the distance of each unit from the equilibrium efficiency frontier, a directed distance function is used. For the unit under evaluation$o$ the deterministic equilibrium, the model is formulated as follows:

$$\begin{aligned} &\max \beta \\ &\text{s.t.} \sum_{j=1}^{n} \lambda_j X_j \leq X_o - \beta g_x \\ &\sum_{j=1}^{n} \lambda_j Y_j \geq Y_o + \beta g_y \\ &\sum_{j=1}^{n} \lambda_j b_j \leq b_o - \beta g_b \\ &\sum_{j=1}^{n} \lambda_j = 1 \\ &\sum_{j=1}^{n} \lambda_j b_j = B_k^{tot} \\ &\lambda_j \geq 0, j = 1, \dots, n \end{aligned} \tag{23}$$

In this model, the direction vector is chosen proportionally to the observed values:

$$g = (-x_o, y_o, -b_o) \tag{24}$$

The optimal value β_o represents the equilibrium directional inefficiency of the unit, and the equilibrium efficiency index is defined as follows:

$$\theta_o = \frac{1}{1 + \beta_o} \tag{25}$$

Due to the uncertainty in operational data, each data point is assumed to be within a bounded interval around its nominal value:

$$x_{ij} \in [x_{ij} - \Delta_{ij}^{x}, x_{ij} + \Delta_{ij}^{x}] \quad (26)$$

$$y_{rj} \in [y_{rj} - \Delta_{rj}^{y}, y_{rj} + \Delta_{rj}^{y}] \quad (27)$$

$$b_{kj} \in [b_{kj} - \Delta_{kj}^{b}, b_{kj} + \Delta_{kj}^{b}] \quad (28)$$

To strengthen the model from the Bertsimas and Sim [10] framework with budget uncertainty, Γ. For example, the input constraint $\sum_{j=1}^{n} b_{kj} = \bar{B}_k, k = 1, \dots, q,$ in the robust case is rewritten as follows:

$$\sum_{j=1}^{n} \lambda_j x_{ij} + \Gamma_i \rho_i + \sum_{j=1}^{n} p_{ij} \leq x_{io} - \beta g_i^{x}, \quad (29)$$

Where

$$p_{ij} \geq \lambda_j \Delta_{ij}^{x} - \rho_i \quad (30)$$

$$p_{ij} \geq 0, \rho_i \geq 0 \quad (31)$$

The corresponding constraints for desirable and undesirable outputs are similarly robustified. Finally, the robust equilibrium efficiency index for each unit is defined as follows:

$$\theta_o^R = \frac{1}{1 + \beta_o^R}, \quad (32)$$

where β_o^R the optimal value of the robust directional inefficiency is. The value $\theta_o^R = 1$ indicates that the unit is on the robust equilibrium efficiency frontier, and values less than one indicate that the unit is far from the system efficiency frontier in the presence of uncertainty and the constraint of a constant sum of pollutants.

Figs. 1-3 show the efficiency robust equilibrium of DMUs and the trend of θ_o^R by ranks and relation to its with β_o^R and the efficiency Trend of by ranks of DMUs respectively.

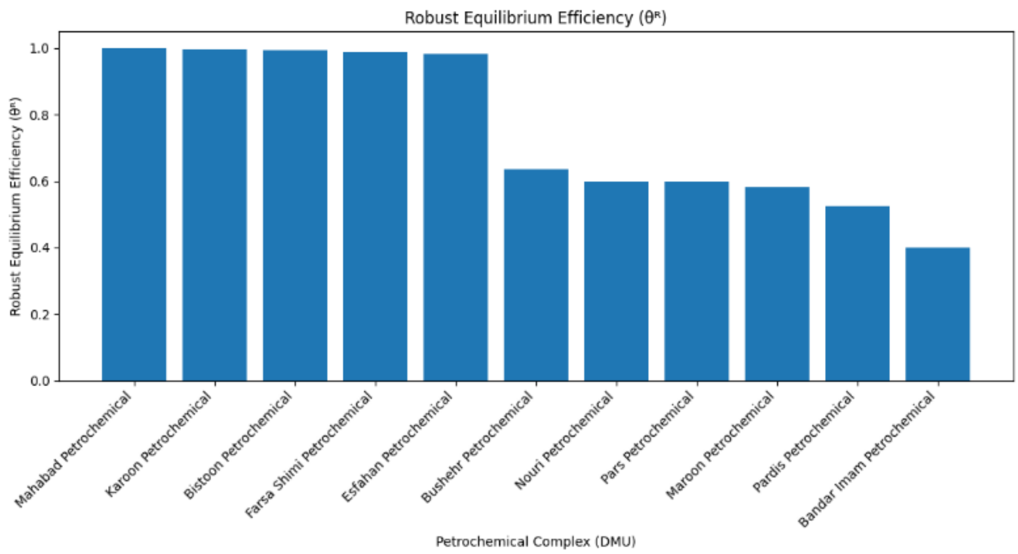

Fig. 1. The efficiency robust equilibrium.

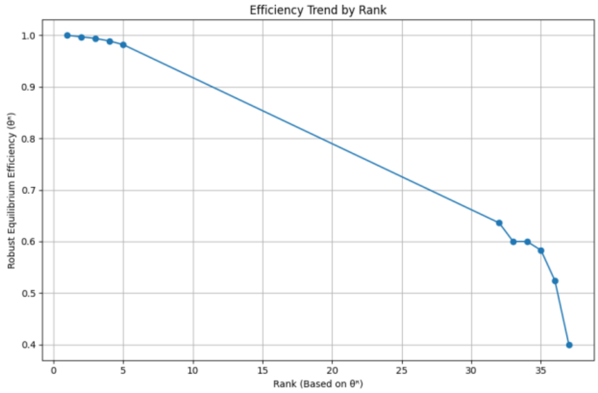

Fig. 2. Trend of θ_o^R by ranks and relation to its with β_o^R.

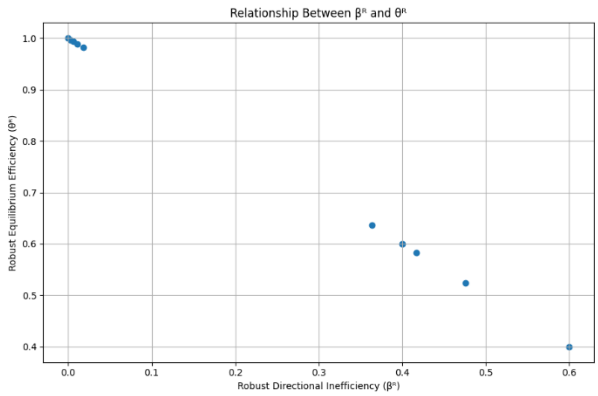

Fig. 3. Efficiency trend by ranks of DMUs.

8 | Modeling Data Uncertainty

The data are assumed to have bounded uncertainty:

$$x_{ij} = \hat{x}_{ij} + \xi_{ij}^x, |\ \xi_{ij}^x\ | \leq \Delta x_{ij} \tag{33}$$

$$y_{rj} = \hat{y}_{rj} + \xi_{rj}^y, |\ \xi_{rj}^y\ | \leq \Delta y_{rj} \tag{34}$$

$$b_{kj} = \hat{b}_{kj} + \xi_{kj}^b, |\ \xi_{kj}^b\ | \leq \Delta b_{kj} \tag{35}$$

Using the uncertainty budget Γ. The model constraints are rewritten robustly, for example, the input constraint:

$$\sum_{j=1}^{n} \lambda_j \hat{x}_{ij} + \Gamma_i^x z_i^x + \sum_{j=1}^{n} p_{ij}^x \leq x_{io} - \beta g_{x_i} \tag{36}$$

$$p_{ij}^x \geq \Delta x_{ij} \lambda_j - z_i^x, p_{ij}^x, z_i^x \geq 0 \tag{37}$$

The corresponding constraints for desirable and undesirable outputs are similarly robustified. After solving the model for each DMU, we define the robust equilibrium efficiency index as follows.

$$\theta_o^R = 1 - \beta_o^R, 0 \leq \theta_o^R \leq 1 \tag{38}$$

8.1|Running a Numerical Example of the Model

Table Specifications of inputs, desired output, and undesirable output CO_2 emissions. Shows six selected petrochemical complexes. *Table 2* shows a numerical example of model implementation based on real data from petrochemical complexes.

Table 2. Numerical example of model implementation based on real data from petrochemical complexes.

CO_2 (Tons)	Main Production (Tons)	Energy (MWh)	Feed (Tons)	Petrochemical Complex
3980500	6634166	5307322	6899524	Bandar Imam Petrochemica
3196908	5328180	4262535	5541300	Pardis Petrochemical
2826880	4711467	3769166	4899919	Maroon Petrochemical
2720863	4534772	3627810	4716157	Pars Petrochemical
2719649	4532748	3626191	4714052	Noori Petrochemical (Borzoyeh)
2496199	4160332	3328259	4326740	Bushehr Petrochemical

In this example, the total CO_2 emissions at the system level are equal to the sum of the values reported in the table below. A robust equilibrium model based on the directed distance function was solved for the selected units. The results showed that complexes with a more favorable ratio of production to pollutant emissions have a smaller directed distance from the equilibrium efficiency frontier, while units with higher pollution intensity show more potential for improving performance within the framework of the constant sum constraint. Also, applying the uncertainty budget reduced the efficiency index values compared to the deterministic case and increased the stability of the results.

8.2|Numerical Implementation of the Proposed Model and Case Study: Petrochemical Industry Performance Evaluation

In order to demonstrate the applicability and efficiency of the proposed model for determining the robust equilibrium efficiency boundary based on the DDF, in this section, a numerical implementation and a case study are performed on a set of Iranian petrochemical complexes. This empirical analysis aims to show how the proposed model can simultaneously evaluate the economic and environmental performance of complexes under conditions where, on the one hand, undesirable outputs are subject to a constant sum constraint at the system level and, on the other hand, operational data are associated with uncertainty. In addition, the obtained results are interpreted analytically to reveal the superiority of the proposed framework over classical DEA models and deterministic equilibrium versions in terms of separability, realism, and stability of the results. In this study, active petrochemical complexes are considered as DMU. These complexes have been selected in such a way that, from the perspective of industrial nature, product type, and overall structure of the production process, there is relative homogeneity among the units in order to meet the assumptions of the DEA method. The data used are extracted from an Excel file collected for the year and include a set of inputs, desirable outputs, and undesirable 1403 environmental outputs.

Based on the nature of the production processes in the petrochemical industry and the availability and reliability of the data, the main inputs of the model were considered to be feedstock consumption (tons) and energy consumption (MWh). The desired output was defined as the main production rate (tons), and the undesirable output was defined as the amount of carbon dioxide (CO_2) emissions. *Table 3* shows an example of actual.

Table 3. Example from real data input and output complexes 1403 petrochemicals (year 1403).

CO_2 (Tons)	Main Production (Tons)	Energy (MWh)	Feed (Tons)	Petrochemical Complex
3980500	6634166	5307322	6899524	Bandar Imam Petrochemica
3196908	5328180	4262535	5541300	Pardis Petrochemical
2826880	4711467	3769166	4899919	Maroon Petrochemical
2720863	4534772	3627810	4716157	Pars Petrochemical
2719649	4532748	3626191	4714052	Noori Petrochemical (Borzoyeh)
2496199	4160332	3328259	4326740	Bushehr Petrochemical

In the framework of the equilibrium problem, it is assumed that the undesirable outcome, CO_2 emissions, has a constant sum property at the system level; this means that the total permissible emission ceiling at the industry level is fixed and any increase in emissions in one complex must be offset by a corresponding reduction in other complexes. This limit is applied as follows:

$$\sum_{j=1}^{40} b_j = \bar{B},$$

where is the total b_j CO_2 emission of the complex j and $\bar{B}$ is the total reported emissions for all complexes under consideration. On the other hand, in order to reflect the reality of industrial data, it is assumed that the observed values of inputs and outputs are not certain and fluctuate in a range around the nominal value. In this numerical implementation, the uncertainty range for all data is 5 bounded and equal to The percentage was considered around the nominal values; in other words:

$\Delta x = \Delta y = \Delta b = 5\%$. After defining the data, applying the constant sum constraint, and considering uncertainty, the robust equilibrium model based on the DDF was solved for all β^R petrochemical complexes. The main output of the model includes the 40, and the robust equilibrium efficiency index, robust directional inefficiency value θ^R which $\theta^R = 1 - \beta^R$ are calculated according to the relationship. The complete results of the model solution and the ranking of the complexes are presented and analyzed in detail in the fourth section. Analysis of the results shows that complexes with a more favorable ratio of production to emissions experience lower amounts of robust directional inefficiency and are closer to the equilibrium efficiency frontier, while units with higher emission intensity achieve lower robust equilibrium efficiency even when consuming less of some inputs. This finding clearly shows that in environments with macro-environmental constraints, simply reducing inputs does not necessarily mean higher efficiency and that emissions and system constraints need to be considered simultaneously in the evaluation process.

In summary, the present numerical implementation based on real data from petrochemical industries shows that the proposed framework is able to evaluate the performance of complexes in a way that both the environmental constraints resulting from the fixed sum of pollutants and the uncertainty of operational data are considered simultaneously. From this perspective, the presented model provides a more realistic, stable, and reliable assessment of environmental-economic performance than classical DEA and deterministic equilibrium models. The industries' petrochemicals provide does and can basis for decision management, and a policy should be put in place at the industry level.

8.3 | Implementing a Robust Equilibrium Model and Analyzing the Results

To empirically evaluate the proposed framework, a robust equilibrium model of active petrochemical 40 based on the directed distance function was implemented on complexes. As explained in the previous chapter, this implementation aimed to simultaneously measure economic and environmental performance. Complexes on conditions are that from one-sided output undesirable CO_2 emissions have a constant sum property at the system level, and on the other hand, operational data face bounded uncertainty. In this framework, each petrochemical complex $DMUj$ was considered as a decision-making unit using the input vector $x_j = (x_{1j}, x_{2j})$ Including feed consumed) tons (and energy consumed (MWh) desired output, $y_j = (y_{1j})$ equal to the original production (tons), and the undesirable output $b_j = (b_{1j})$ produces equivalent to CO_2 emissions (tons).

petrochemical 40 The results of implementing the robust equilibrium model for complexes are presented in *Table 5*. In this table, for each DMU, the value of the robust directional inefficiency β^R the robust equilibrium efficiency index, θ^R and the final rank is reported. As previously stated, the robust equilibrium efficiency index $\theta_j^R = 1 - \beta_j^R$ is calculated based on the relationship and is the basis for ranking the units; this means that the θ_j^R larger the value, the closer the unit is to the robust equilibrium efficiency frontier.

Table 4. Active DMU operator for 1403 years.

New Rank	Petrochemical Complex	θ^R	β^R
1	Mahabad	1	0
2	Karun	0.997	0.003
3	Bison	0.994	0.006
4	Farsa Chemistry	0.989	0.011
5	Isfahan	0.982	0.018
6	Khorasan	0.976	0.024
7	Tabriz	0.969	0.031
8	Arvand	0.958	0.042
9	Jam	0.942	0.058
10	Zagros	1	0
10	Zagros	0.927	0.073
11	Shiraz	0.911	0.089
12	Amir Kabir	0.896	0.104
13	Tulip	0.882	0.118
14	Kermanshah	0.864	0.136
15	Razi	0.849	0.151
16	Technologists	0.832	0.168
17	Loudmouths	0.816	0.184
18	Kazerun	0.799	0.201
19	Farabi	0.781	0.219
20	Abadan	0.746	0.254
21	Ilam	0.728	0.272
22	AryaSasol	0.711	0.289
23	Shazand	0.693	0.307
24	Dehloran	0.675	0.325
25	Persian Gulf	0.659	0.341
26	When	0.648	0.352
27	Lorestan	0.639	0.361
28	Suleiman Mosque	0.637	0.363
29	Buali	0.637	0.363
30	Kavian	0.636	0.364
31	Bushehr	0.6	0.4
32	Optical	0.6	0.4
33	Pars	0.583	0.417
34	Maroon	0.524	0.476
35	Campus	0.4	0.6
36	Imam Port	0.4	0.6

The active complexes operating in 36. The statistical population includes elimination of units under construction, resulting in the formation of a robust equilibrium efficiency frontier based solely on actual performance data, which increases the validity of the results and the robustness of policy inferences. In this structure, only one complex) Mahabad (has achieved full equilibrium efficiency of the units, has 42%, and about θ^R less than 0.70, indicating a significant distance of the industry from the robust equilibrium state within the framework of the constant total emission constraint. In contrast, complexes such as Bandar-e-Imam, Pardis, and Maroon have relatively high values of β^R indicating a significant distance from the sustainable equilibrium efficiency, frontier. Specifically, the value θ^R= 0.400 for Bandar-e-Imam indicates that the percent of its potential equilibrium efficiency, and, from a 40 complex, has achieved only a systemic perspective, requires serious reforms in the production structure or reduction of pollution intensity. Importantly, this ranking does not simply reflect technical efficiency at the unit level, but is the result of the interaction of units within the framework of the constant sum of emissions constraint, and taking into account data uncertainty.

One of the notable findings of *Table 5* is that the range of variations θ^R is relatively wide, ranging from 1.000 to 0.400. This dispersion indicates that the robust equilibrium model has been able to provide adequate resolution between complexes and avoids the problem of over-identification of efficient units observed in some classical models. In addition, the closeness of the values of some units, such as Noori and Pars, both with θ^R= 0.600) It indicates the relative similarity of their functional structure in an equilibrium framework. Overall, the results in *Table 5* show that the proposed model is able to make an accurate and meaningful distinction between the performance of complexes, especially in situations where pollution and environmental constraints play a decisive role in defining efficiency. This distinction is the result of the simultaneous combination of the constant sum constraint, the non-radial structure of the directed distance function, and the robust approach to data uncertainty, and it represents the analytical superiority of the robust equilibrium model over the classical deterministic and independent frameworks. *Figs. 3* and *4* show the equilibrium efficiency and sensitivity analysis to different levels of uncertainty (Γ), respectively.

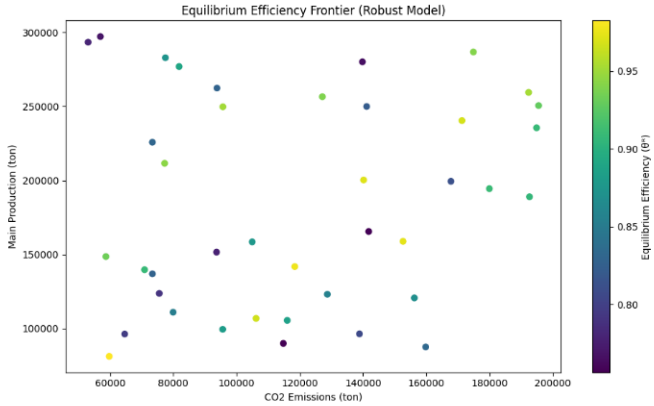

Fig. 3. The equilibrium efficiency.

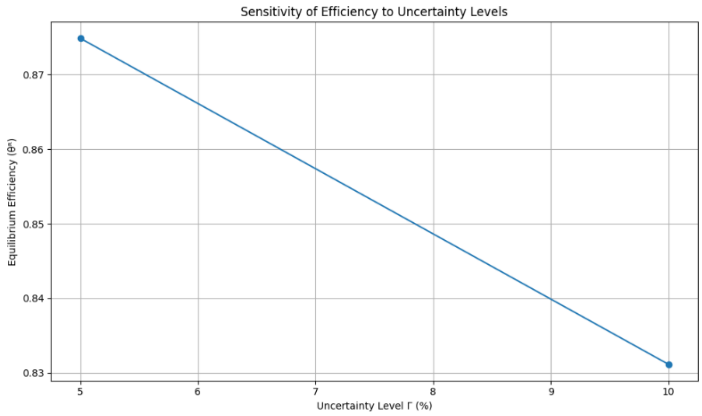

Fig. 4. The sensitivity analysis to different levels of uncertainty (Γ), respectively.

8.4 | Analysis of the Results of Implementing the Robust Equilibrium

The implementation of a robust equilibrium model based on the directional distance shows that the efficiency structure of 1403 active complexes operating in 36 functions in the petrochemical industry under the condition of a constant total CO_2 emission constraint is much more complex than that observed in classical DEA models. In this framework, the efficiency of each complex is not only a function of its independent technical performance, but also the result of the systemic interaction of the entire industry under a common environmental constraint; therefore, the robust equilibrium efficiency index (θ^R) reflects a system-equilibrium efficiency is and No simply efficiency technical Conventional. θ^R changes from 1.000 to 0.400 in θ^R indicates significant structural heterogeneity among the complexes. This dispersion indicates that some units have achieved a level of alignment between economic production and emission intensity, while others are still a significant distance from the resilient equilibrium frontier. Only the Mahabad complex has a value of $\beta^R = 0$ is located exactly on the equilibrium efficiency frontier; this means that within the current production and emission constraints framework, there is no possibility of simultaneous improvement in increasing the desired output and reducing emissions for this unit.

In contrast, complexes such as Bandar Imam, Pardis, and Maroon with lower values of θ^R are a significant distance from the frontier. In particular, the value of θ^R=0.400 of its % 40 for Bandar Imam indicates that this complex has only achieved equilibrium potential capacity and, from a systemic perspective, requires structural reforms in energy intensity, feedstock composition, or production technology. Importantly, the robust equilibrium model, unlike classical DEA models, has prevented over-identification of efficient units. In traditional DEA frameworks, due to the deterministic nature of the data and the lack of a systemic diffusion constraint, several units are usually introduced as efficient; however, in the present proposed structure, which simultaneously considers the diffusion dependence between units and the uncertainty of the data, only one complex has achieved full efficiency. This indicates the high resolution of the model and its ability to make a more precise distinction between the actual performance of complexes. In addition, the closeness of the θ^R Values of some complexes, such as Noori and Pars, indicate the relative similarity of their functional structure in the equilibrium framework and allow for the policy clustering of these units. Sensitivity analysis to different levels of uncertainty, Γ, also confirms the robustness of the proposed model. As the uncertainty level increases from Γ=0 percent, the 10 and 5 to values of θ^R decrease uniformly and predictably, indicating that efficiency evaluation becomes more conservative under higher uncertainty conditions.

However, the relative ranking of complexes experiences limited changes. This rank stability indicates that the efficiency structure of the industry is stable to limited data fluctuations, and the robust equilibrium model does not suffer from irrational rank jumps. If an increase in Γ led to drastic changes in the ranking, it could be a sign of structural fragility of the industry; however, the present results indicate a level of systemic robustness within the equilibrium evaluation framework.

8.5 | Comparison of the Results of the Proposed Model with Reference Models

To evaluate the resolution and realism of the proposed robust equilibrium model, 36, the results of three different DEA frameworks, were compared on active petrochemical complexes. These frameworks include: 1) a classical input-driven DEA model without considering undesirable outputs with a constant sum property, 2) a deterministic equilibrium model with a constant sum constraint on CO_2 emissions and without robustness, and 3) a robust equilibrium model based on a directed distance function (the proposed model). *Table 5* presents the number of efficient units identified in each of these models.

Table 5. Comparison of various number units kara in models.

Evaluation Model	Number of Efficient Units	Percentage of Total Units
Classical DEA	7	19%
Deterministic equilibrium	3	8%
Robust equilibrium (proposed)	1	3%

Based on the results of *Table 5* in the classical DEA model, complexes (equivalent to 7 of the units) have been identified as efficient. This relatively high value can be % 19 to attributed to the assumption of independence of the DMUs and the lack of consideration of the systemic constraint of a constant sum of CO_2 emissions. framework, each complex has the possibility of being on the efficiency frontier without considering the impact of its performance on other units and without considering the macro-environmental constraints.

By applying a fixed sum constraint in the deterministic equilibrium model, the number. This reduction indicates that in the (% 8) complexes, 3 of the efficient units are reduced to the presence of a systemic emission constraint, the possibility of simultaneous efficiency for multiple units is reduced, and the evaluation structure is shifted from a unit-based to a system-based equilibrium Change Does. In actuality, apart from functionalities identified in the model classic caused by ignoring dependency mutual units, it is finally, considering the uncertainty of the data in the robust equilibrium model, only one complex (Mahabad) is on the robust equilibrium efficiency frontier. This result shows that some units that are known to be efficient in the deterministic models are not stable against data fluctuations, and their efficiency is fragile. Applying the robust approach causes the efficiency frontier to be more compact, the number of efficient units to decrease, and the evaluation to become more rigorous. The units in the 7. In general, the gradual decrease in the number of efficient units from the unit in the robust equilibrium model indicates increased resolution 1 classical model to greater realism, and higher stability of the proposed model in assessing environmental economic performance. Complexes Petrochemicals This is a Comparison Numerical Badge Gives that Model Equilibrium Resistant Not only Dependency Systemic Caused From Adverb Total Fixed particle for direct object Consideration Does, Rather From Identification Functionalities Appearance and Unstable Also Prevent It seems.

9 | Management and Policy Applications

The proposed model can help complex managers identify performance weaknesses and improve resilience to data changes. In addition, more accurate predictions for environmental policies and optimal allocation of emission permits are prominent features of this model. These conclusions indicate that the robust equilibrium model can be used as a valid tool for managerial decision-making and macro-policy design in the petrochemical industry.

9.1 | Management Implications for Petrochemical Complexes

The results of implementing a robust equilibrium model based on the DDF on 40 petrochemical complexes show that some of the complexes that are identified as efficient units within the framework of classical DEA models have a significant distance to the boundary on the robust equilibrium efficiency frontier. This difference is due to the fact that classical models usually ignore the interdependence of units and environmental constraints at the industry level and evaluate efficiency solely based on nominal and independent data of each unit. From a managerial perspective, this finding is important because it shows that the "apparent efficiency" of some complexes does not necessarily mean stable and reliable performance in real environments. By taking into account the constraint of a constant sum of pollutants and the uncertainty of operational data, the proposed model reveals the hidden weaknesses of the complexes' performance and provides a more realistic picture of the position of each unit in the overall industry structure. Thus, instead of relying on the optimistic results of classical models, petrochemical complex managers can use the robust equilibrium efficiency index as a more accurate basis for evaluating performance and designing improvement programs.

The robust directional inefficiency index extracted from the proposed model provides specific information about the potential of each complex for performance improvement. This index shows the extent to which each unit can simultaneously optimize input consumption, increase production levels, and reduce pollutant emissions without violating environmental constraints at the industry level. Such an approach helps managers move away from focusing on single-dimensional improvements and adopt multidimensional, realistic, and regulatory-compliant strategies.

9.2 | Pollutant Management and Optimal Allocation of Emission Permits

One of the most important advantages of the proposed model is its ability to analyze the optimal allocation of pollutants among petrochemical complexes in the presence of a total emission cap. In environments where total pollutant emissions are limited at the industry level, deciding how to allocate emission permits to different units plays a decisive role in improving environmental and economic efficiency.

The results of the model implementation show that uniform or historical pattern-based allocation of emission permits does not necessarily lead to improved efficiency of the entire industry. In contrast, the proposed robust equilibrium framework allows for emission permits to be allocated based on the actual potential of each complex to reduce emissions and the implicit costs of this reduction. Complexes that have a higher capacity to reduce emissions in terms of technology, production structure, or pollution intensity can take on a greater share in achieving the environmental goals of the industry, while other units benefit from this capacity.

From a policy perspective, the proposed model can be used as a supporting tool for designing market-based mechanisms, such as an emissions trading system. The model results show that such mechanisms, if designed based on robust equilibrium efficiency indices, can increase the overall efficiency of the petrochemical industry and reduce the costs of achieving environmental goals while maintaining the total emission ceiling.

9.3 | The Role of Uncertainty in Managerial Decision-Making

One of the key findings of the research is the significant difference between the results of the deterministic models and the robust equilibrium model. The results show that in deterministic frameworks, the ranking of complexes is sensitive to small changes in input and output data and may fluctuate significantly in real conditions. Such instability can lead managers to make decisions based on fragile and unreliable information.

In contrast, the robust equilibrium model presented in this study provides more stable and reliable results by considering data uncertainty in the evaluation process. From a managerial perspective, this feature means reducing decision-making risk. Managers can rely more confidently on the results of the model and design performance improvement, investment, or technology modernization programs based on indicators that are robust to data fluctuations.

9.4 | Interpretation of Results at the Industry Level and Macro Policymaking

At the macro level of the petrochemical industry, the results of this study show that unit-based approaches to performance assessment alone are not able to ensure sustainable improvement in environmental performance. The existence of a fixed total of pollutants constraint requires that policymaking be carried out at the industry level and by considering the interaction and interdependence of complexes. The proposed model, by providing a common equilibrium efficiency frontier, enables such a view. From a public policy perspective, the framework can help regulators assess the implications of environmental policies on industry efficiency before they are implemented. For example, it can be used to examine how stricter emission caps affect the distribution of efficiency in an industry and which industries are most affected by these policies. This capability allows for the design of gradual, targeted, and evidence-based policies and prevents unwanted shocks to the industry.

Overall, the research results show that the proposed model is not only theoretically sound but also applicable from a managerial and policy perspective. This model can help petrochemical complex managers identify paths for performance improvement and, at the same time, provide an analytical tool for policymakers to make environmental decisions based on sustainable performance assessments. Hence, the proposed framework can play an effective role in promoting the productivity and environmental sustainability of the petrochemical industry.

10 | Conclusion and Suggestions

The main objective of this research was to provide a comprehensive analytical framework for evaluating the performance of petrochemical industries in conditions where undesirable outputs have a constant sum property and operational data are associated with uncertainty. To achieve this goal, a DEA-based model was developed in which the equilibrium efficiency frontier, the directed distance function, and the robust optimization approach were integrated. This combination allowed for the simultaneous analysis of the economic and environmental dimensions of the performance of petrochemical complexes in a more realistic environment.

The theoretical and experimental results of the research showed that ignoring the interdependence of petrochemical units due to the limitation of the fixed sum of pollutants can lead to optimistic and unrealistic assessments of performance. By creating a common equilibrium efficiency boundary, the proposed model explicitly considers this dependence in the evaluation process and provides a fairer comparison between complexes. On the other hand, the use of the directed distance function caused the inefficiency of the units to be measured in a non-radial manner and in proportion to the true nature of petrochemical processes, something that is not accurately reflected in classical radial models.

In addition, the results of the case study showed that the use of a robust optimization approach increases the stability of the efficiency frontier and performance evaluation results against data fluctuations. Comparison of the results of the proposed model with classical DEA models and deterministic equilibrium models indicates that the proposed framework provides a more realistic and reliable assessment of environmental-economic performance. The industries' petrochemicals presentation is given. this feature, usability results research particle for direct object for decision-making, management, and policymaking, environmental increase gives.

10.1 | Research Limitations

Despite the achievements of the present study, there are some limitations. First, the case study was conducted based on available data, and the quality of the results depends to some extent on the accuracy of the input and output data. In this regard, more data may be needed in different industries to achieve more comprehensive results, including market conditions, price fluctuations, and other economic factors, in addition to the amount of inputs and outputs. Second, in this study, data uncertainty is modeled in the form of bounded sets, while in some applications, there may be more complex structures of uncertainty that require more accurate modeling. For example, in some industries, multiple and dependent effects may affect the efficiency results in a nonlinear manner. Third, the presented model is static in nature, and the temporal changes in the performance of petrochemical complexes are not directly included in it. Therefore, in times when rapid changes occur in the market or environmental conditions, the proposed model may need to be updated or adjusted.

10.2 | Suggestions for Future Research

Based on the above limitations and the results obtained, several directions for future research are suggested:

I. Development of the proposed model in the dynamic DEA framework: this model can enable the study of changes in the efficiency of petrochemical complexes over time. Efficiency assessment based on dynamic models, especially in environments that are constantly changing, can help to more realistically simulate the performance of units over a period of time.

II. Combining the presented approach with other uncertainty modeling methods, Such as fuzzy or stochastic approaches, can help to more accurately analyze uncertain conditions. For example, using fuzzy methods can allow the model to achieve more accurate and comprehensive results while taking into account uncertainties arising from input data.

III. Applying the proposed model to other polluting industries, such as steel, cement, and power plants, can test the generalizability of the research framework and examine its applicability to other industries. In fact, using the proposed model in industries that have similarities to petrochemicals can show whether the model is also effective for evaluating performance in these industries.

IV. Integrating model results with multi-criteria decision-making tools: this can help managers and policymakers in choosing optimal environmental and economic strategies. The use of multi-criteria decision-making tools can provide more optimal solutions for managing resources and pollutants that may not be visible in one-dimensional models.

V. Increasing accuracy in modeling environmental fluctuations and complexities: by using methods based on systems analysis and data enhancement, researchers can consider greater complexities of the industrial environment and its potential fluctuations in the model, so that the final result is more accurate and reliable.

In summary, by presenting an integrated model for determining the robust equilibrium efficiency frontier, the present study has taken a step towards improving the performance assessment of petrochemical industries under real and uncertain conditions and can be a basis for developing future studies in the field of efficiency, sustainability, and environmental policy-making of energy-intensive industries. This model is not only theoretically valid but can also be used in practice as an efficient tool for managerial and policy decisions in the petrochemical industry and other similar industries.

Conflict of Interest

The authors declare no conflict of interest.

Data Availability

All data are included in the text.

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References

[1] Yang, M., An, Q., Hu, D., & Liang, L. (2021). Performance evaluation of China's industry: A generalized equilibrium data envelopment analysis approach with fixed-sum undesirable output. *INFOR: information systems and operational research*, *59*(2), 290–308. https://doi.org/10.1080/03155986.2021.1881360

[2] Aparicio, J., Pastor, J. T., & Zofio, J. L. (2013). On the inconsistency of the Malmquist–Luenberger index. *European journal of operational research*, *229*(3), 738–742. https://doi.org/10.1016/j.ejor.2013.03.031

[3] Zhu, Q., Song, M., & Wu, J. (2020). Extended secondary goal approach for common equilibrium efficient frontier selection in DEA with fixed-sum outputs. *Computers & industrial engineering*, *144*, 106483. https://doi.org/10.1016/j.cie.2020.106483

[4] Zhou, P., Ang, B. W., & Poh, K. L. (2008). Measuring environmental performance under different environmental DEA technologies. *Energy economics*, *30*(1), 1–14. https://doi.org/10.1016/j.eneco.2006.05.001

[5] Färe, R., Grosskopf, S., Lovell, C. A. K., & Pasurka, C. (1989). Multilateral productivity comparisons when some outputs are undesirable: a nonparametric approach. *The review of economics and statistics*, 90–98.

[6] Zhu, Q., Wu, J., Song, M., & Liang, L. (2017). A unique equilibrium efficient frontier with fixed-sum outputs in data envelopment analysis. *Journal of the operational research society*, *68*(12), 1483–1490. 10.1057/s41274-017-0181-z

[7] Chambers, R. G., Chung, Y., & Färe, R. (1996). Benefit and distance functions. *Journal of economic theory*, *70*(2), 407–419. https://doi.org/10.1006/jeth.1996.0096

[8] Sueyoshi, T., & Goto, M. (2015). DEA environmental assessment in time horizon: Radial approach for Malmquist index measurement on petroleum companies. *Energy economics*, *51*, 329–345. https://doi.org/10.1016/j.eneco.2015.07.010

[9] Arabmaldar, A., Sahoo, B. K., & Ghiyasi, M. (2023). A generalized robust data envelopment analysis model based on directional distance function. *European journal of operational research*, *311*(2), 617–632. https://doi.org/10.1016/j.ejor.2023.05.005

[10] Bertsimas, D., & Sim, M. (2004). The price of robustness. *Operations research*, *52*(1), 35–53. https://doi.org/10.1287/opre.1030.0065

[11] Shokouhi, A. H., Hatami-Marbini, A., Tavana, M., & Saati, S. (2010). A robust optimization approach for imprecise data envelopment analysis. *Computers & industrial engineering*, *59*(3), 387–397. https://doi.org/10.1016/j.cie.2010.05.011

[12] Charnes, A., Cooper, W. W., & Rhodes, E. (1978). Measuring the efficiency of decision making units. *European journal of operational research*, *2*(6), 429–444. https://doi.org/10.1016/0377-2217(78)90138-8

[13] Zhu, Q., Li, X., Li, F., Wu, J., & Zhou, D. (2020). Energy and environmental efficiency of China's transportation sectors under the constraints of energy consumption and environmental pollutions. *Energy economics*, *89*, 104817. https://doi.org/10.1016/j.eneco.2020.104817

[14] Seiford, L. M., & Zhu, J. (2002). Modeling undesirable factors in efficiency evaluation. *European journal of operational research*, *142*(1), 16–20. https://doi.org/10.1016/S0377-2217(01)00293-4

[15] Tone, K. (2001). A slacks-based measure of efficiency in data envelopment analysis. *European journal of operational research*, *130*(3), 498–509. https://doi.org/10.1016/S0377-2217(99)00407-5

[16] Lins, M. P. E., Gomes, E. G., Soares de Mello, J. C. C. B., & Soares de Mello, A. J. R. (2003). Olympic ranking based on a zero sum gains DEA model. *European journal of operational research*, *148*(2), 312–322. https://doi.org/10.1016/S0377-2217(02)00687-2

[17] Wang, W., Xu, F., Chu, J., Dong, Y., & Yuan, Z. (2025). Determining the equilibrium efficient frontier by proportional frontier shifting for data envelopment analysis with fixed-sum outputs. *Omega*, *130*, 103174. https://doi.org/10.1016/j.omega.2024.103174

[18] Gomes, E. G., & Lins, M. P. E. (2008). Modelling undesirable outputs with zero sum gains data envelopment analysis models. *Journal of the operational research society*, *59*(5), 616–623. 10.1057/palgrave.jors.2602384

[19] Yang, F., Wu, D., Liang, L., Bi, G., & Wu, D. D. (2011). Supply chain DEA: Production possibility set and performance evaluation model. *Annals of operations research*, *185*(1), 195–211. 10.1007/s10479-008-0511-2

[20] Chung, Y. H., Färe, R., & Grosskopf, S. (1997). Productivity and undesirable outputs: a directional distance function approach. *Journal of environmental management*, *51*(3), 229–240. https://doi.org/10.1006/jema.1997.0146

[21] Yang, M., Li, Y. J., & Liang, L. (2015). A generalized equilibrium efficient frontier data envelopment analysis approach for evaluating DMUs with fixed-sum outputs. *European journal of operational research*, *246*(1), 209–217. https://doi.org/10.1016/j.ejor.2015.04.023

[22] Ben-Tal, A., & Nemirovski, A. (2002). Robust optimization--methodology and applications. *Mathematical programming*, *92*(3), 453–480. 10.1007/s101070100286

[23] Cooper, W. W., Seiford, L. M., Tone, K. (2007). *Data envelopment analysis: A comprehensive text with models, applications, references and DEA-solver software* (Vol. 2). Springer. https://doi.org/10.1007/978-0-387-45283-8