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The Readout of Merton's Problem on Infinite Horizon - Stochastic Optimal Control Modelling

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Abstract

Stochastic modeling is of the utmost importance in a world dominated by uncertainty. Many problems in economics, finance, and actuarial science naturally require decision-makers to undertake choices in stochastic environments. Classical methods for solving infinite horizon Stochastic Optimal Control Problems (SOCPs) primarily focus on deriving the solution by defining the value function through dynamic programming and the Hamilton-Jacobi-Bellman equation. However, obtaining a closed-form solution is generally challenging. This article proposes a hybrid method for solving SOCPs to address this issue and identify an optimal trajectory and control. This novel approach integrates the Multi-Step Stochastic Differential Transform Method (MSDTM) with an approximation technique that solves infinite horizon problems by leveraging a finite horizon. An applicable example diagram of the types of instances created from the simulation of the described approach and infinite horizon stochastic optimal control problem from management science is provided to show the method's effectiveness and efficiency, particularly in comparison with existing approaches.

Keywords: Stochastic optimal control problems, Dynamic programming, Hamilton-Jacobi-Bellman equation, Merton's portfolio problem.

1 | Introduction

Banks, investment funds, and insurance companies are examples of investors that invest money in the financial markets. They want to make as much money as possible on their investments, but any serious investor must also consider the risk involved. To a certain degree, an investor is risk-averse, i.e., the investor

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is reluctant to invest in an asset with high potential if this means that the risk of losing money is also high. Such investors aim to maximize the expected returns on their investments while at the same time limiting the risk involved. One way of modeling such behaviors is through stochastic control theory and the maximization of expected utility functions.

Stochastic Optimal Control Problems (SOCPs) frequently occur in many science branches, especially economics and finance. Stochastic differential equations have become the standard models for financial quantities such as asset prices, interest rates, and their derivatives [1], [2]. Three major approaches in SOCOs can be differentiated: Dynamic programming, duality, and the maximum principle [3]. Dynamic programming obtains the Hamilton-Jacobi-Bellman equation using Bellman's optimality principle, which characterizes the value function. In fact, in the case of continuous optimal control problems, the dynamic programming technique reduces the optimal control problem to solving a partial differential equation (The Hamilton-Jacobi-Bellman (HJB) equation). Almost all (Just a few special cases) of these equations are difficult to solve analytically [4]. Under some smoothness and regularity assumptions on the solution, it is possible to obtain, at least implicitly, the optimal control. This is the content of the so-called verification theorems in [5]. However, only a few classes of SOCPs admit analytical solutions for the value function in this manner. Besides, finite state Markov chain approximation and finite differences are two classical approaches in solving SOCPs numerically, but analytical and analytical-numerical methods are superior over the numerical methods due to the determination of closed-form solutions [6–8].

The paper is organized as follows: First, we present the control problem and the preliminaries used in the following sections. In Section 3, we review the SDTM approach and its hypotheses. In Section 4, we show how infinite-horizon SOCOs can be solved by studying their finite horizon approximations, and the new hybrid method for solving infinite-horizon SOCPs is explained. To demonstrate the application and efficiency of the new method, Section 5 is devoted to describing and solving one of the famous and useful problems in management, namely the Merton problem. Finally, the conclusion is presented in Section 6.

2 | Problem Statement

In this section, we consider finite-time homogeneous SOCPs of the type.

$$\mathrm{Sup}_{u\in U} E\left(\int_0^\infty e^{-\beta t} F(X(t),u(t))dt\right), \mathrm{S.to}: dx(t) = f\big(X(t),u(t)\big) + \sigma\big(X(t),u(t)\big)dB(t);\ X(0) = x_0, \tag{1}$$

where $u \in U \subseteq R^n$ denotes the set of admissible controls and $x \in X \subseteq R^m$ is a state vector. Furthermore, $F: X \times U \to R$ it is the utility function and $\beta \geq 0$ is the discount rate. In our approach, we have the following limitations, notations, and definitions:

I. We restrict this problem to the one-dimensional SOCPs for simplicity of the exposition;

II. We denote (F_t) with the filtration generated by Brownian motion and assume that f, σ, and F are continuous functions on $S \times U$, that S and U are respectively closed intervals in $\mathcal{R}$ and refer to state and control space;

III. An (F_t) –adapted stochastic process is called a feasible control if almost surely for all t, we have $u(t) \in U$;

IV. We let $D \in C^2(S)$ be all the smooth functions on S with bounded derivatives of all orders, so that it includes the value functions of *Model (1)*.

Definition 1. A feasible control u(.) is admissible, if

The governed equation of *Model (1)*, a stochastic differential equation, admits a unique solution;

For all $\emptyset \in D$ and for all $t > 0$, the Dynkin *Formula (1)* holds and;

$$E(|\emptyset(X^u(t))| < \infty,$$

$$E\left(\int_0^\infty e^{-\beta}|F(X^u(s), u(s))|ds\right) < \infty.$$

3 | Multi-Step Stochastic Differential Transformation Method

One of the effective methods for solving differential equations is the Differential Transformation Method (DTM), which was initially introduced by Zhou [9]. This method obtains an analytical solution as a polynomial based on the Taylor series for differential equations [10]. It has been applied to solving deterministic optimal control problems [11], [12]. In multi-step DTM, a modification of DTM and the convergence of the obtained solution series are improved. Fakharzadeh et al. [13] proposed an efficient and fast approach for the multi-step DTM, a reliable modification of the DTM that enhances the convergence of the series solution. The method provides immediate and visible symbolic terms of analytic solutions. The DTM is developed for stochastic calculus to solve stochastic differential equations [13]. To introduce the tools for subsequent discussions, in this section, we remind some necessary basic definitions and properties, which can mostly be found in [1] and [14].

Definition 2. (Stochastic calculus). For a given probability space (Ω, F, P), a real random variable X defined on this space and satisfies the condition $E(X^2) < +\infty$ is called a second-order random variable (2-r.v.); here E, it denotes the expectation operator. The space L_2 of all the 2-r.v.'s endowed with the norm $\|X\|_2 = \sqrt{E(X^2)}$, is a Banach space [14]. In the probability space (Ω, F, P), a stochastic process $\{X(t): t \in I\}$ I with a closed interval in the real line $\mathbb{R}$ is called a second-order stochastic process (2-s.p.) if for each $t \in I, X(t)$ is a 2-r.v. A sequence of 2-r.v.'s $\{X_n : n \geq 0\}$ is mean square convergent in L_2 to a 2-r.v. X as $n \to \infty$ if

$$\lim_{n\to\infty} \|X_n - X\|_2 = 0.$$

Also if we have a limiting condition such that

$$\lim_{\Delta t \to 0} \left\| \frac{X(t+\Delta t) - X(t)}{\Delta t} - \dot{X}(t) \right\|_2 = 0.$$

Then, 2-s.p $\{\dot{X}(t): t \in I\}$ is the mean square derivative of $\{X(t): t \in I\}$.

Definition 3. Let $k \in \mathbb{N}$ us assume that the 4-s.p. $\{v(t): t \in I\}$ has a mean fourth derivative of order k $t \in I$, which is denoted by $^{(k)}(t)$. The random differential transform of the process $v(t)$ is defined as:

$$V(k) = \frac{1}{k!}\left[\frac{d^k(v(t))}{dt^k}\right]_{t=t_0}, \tag{2}$$

where V the second process is transformed, and $\frac{d}{dt}$ denotes the mean square derivative. The inverse transform V is defined as:

$$v(t) = \sum_{k=0}^{\infty} V(k)(t - t_0)^k, \tag{3}$$

Here, it is formally assumed that the *Model* (3) is uniformly mean fourth convergent in any closed interval into the domain of convergence, and as a result, *Model (2)* and *(3)* are well defined [15]. For implementation purposes, the function v(t) is expressed by a finite series, and *Eq. (3)* can be written as:

$$v(t) \approx \sum_{k=0}^{N} V(k)(t-t_0)^k.$$

The convergence of natural frequency decides that N. For a stochastic differential equation, we assume all the involved stochastic quantities take values in L_2-space and all the stochastic operations are in the mean square sense. We note that $B(t)$ a Gaussian process with a mean zero and a mean fourth continuous process [16]. The Brownian motion $B(t)$ has trajectories belonging to $L^2([0,T])$ almost all events. In this space, the Karhunen-Lèove expansion for Brownian motion takes the form [16]:

$$B(t) = B(t,\omega) = \sum_{i=0}^{\infty} z_i(\omega)\varphi_i(t), 0 \le t \le T,$$

with:

$$\varphi_i(t) = \frac{2\sqrt{2T}}{(2i+1)\pi}\sin\left(\frac{(2i+1)\pi t}{2T}\right).$$

The functions $\varphi_i(t)$'s form a basis of orthogonal functions and { } is a sequence of independent and identically distributed Gaussian random variables [16]. This approach is also quite powerful for simulating paths of processes without independent increments of Brownian motion. By substituting finite terms of the Karhuneun-Loève expansion in *Model (2)*, we have [16]:

$$dx(t) = a(x,t)dt + \sigma(x,t)d\left(\sum_{i=0}^{M} z_i\varphi_i(t)\right), x(0) = x_0. \tag{4}$$

Applying the random DTM from *Model (2)* and using the properties (I)-(IV) in *Model (1)*, one can transfer *Model (4)* into the following algebraic equation:

$$(k+1)X(k+1) = A(k) + \sum_{i=0}^{m}\sum_{j=0}^{k} z_i\,\Psi_i(k)\Sigma(k-j); X(0) = x_0, \tag{5}$$

where $X(k), A(k), \Sigma(k)$, and $\Psi(k)$ are the transformed processes of x, a, σ and the derivative of φ_i, respectively. z_i For instance, one can use the Maple random variable generator (Random tools flavor: Distribution) to simulate.

We remember that [0, T] is the interval over which we want to find the *Model (4)* over t. We apply a multi-step approach to ensure the validity of the mentioned approximations for large T. Our new approach assumes that the interval [0, T] is divided into M subintervals $[t_{i-1}, t_i], i = 1,2,\ldots,M$ with equal step length. Thus, first, we use the SDTM to *Eq. (4)* over the interval $[0, t_1]$, and then at each subinterval $[t_{i-1}, t_i], i \ge 2$, the SDTM is applied to *Eq. (6)* over the interval $[t_{i-1}, t_i]$, where t_0 it is replaced by t_{i-1}. It is necessary to refer to the initial conditions of the above approach as $x_i(t_{i-1}) = x_{i-1}(t_{i-1})$, $i = 2,3,\ldots,M$.. Therefore, the process is repeated to generate a sequence of approximate solutions $x_i(t), i = 1,2,\ldots$, M. The MSDTM obtains the following solution as a piecewise polynomial $[0,T]$:

$$x(t) = \begin{cases} x_1(t), t \in [0, t_1], \\ x_2(t), t \in [t_1, t_2], \\ \vdots \\ x_M(t), t \in [t_{M-1}, t_M]. \end{cases} \tag{6}$$

Proposition 1. Consider the *Model (4)* and assume that all of the conditions of *Theorem 2* are satisfied. Also, by defining $F(X(t),t) = P_n(t)X(t) + Q(t)$, we take $F: S \times T \to L_2$ is continuous and satisfies the m.s. Lipchitz condition. Then, there exists a unique m.s. Solution for any initial condition $x_0 \in L_2$.

Proof: See [13].

This approach aims to extend the application of multi-step DTM to obtain an approximated analytical solution of SOCPs. To this end, we use the HJB equation and the Multi-Step Stochastic Differential Transform Method (MSDTM) to determine the approximated optimal trajectory and optimal control of SOCPs based on the Brownian motion properties in L^2 – spaces and the properties of the Karhunen-Loève expansion.

4 | Approximation Method for Solving Stochastic Optimal Control Problems with an Infinite Horizon

In this section, we review the method of [5], which demonstrated how finite horizon approximations to infinite horizon SOCPs can be used to obtain analytic solutions. This often leads to analytical solutions for the endless horizon SOCPs by studying phase diagrams. First, we consider the truncated stochastic optimal control problem,

$$\mathrm{Sup}_{u\in A}\ E\left(\int_0^T e^{-rt}\, F\big(X(s),u(s)\big)ds\right), \mathrm{S.to}: dX(t) = f\big(X(t),u(t)\big)dt + \sigma\big(X(t),u(t)\big)dW(t). \tag{7}$$

For finite T. We denote its value function as follows:

$$V(t,T,x) = \mathrm{Sup}_{u\in A_T} E(\int_t^T e^{-r(s-t)}\, F(X(s),u(s))ds), \tag{8}$$

and its HJB-equation has the form shown in *Eq. (9)*:

$$\begin{gathered} rV(t,T,x) - V_t(t,T,x) = \mathrm{Sup}_{u\in U}\left\{F(x,u) + V_x(t,T,x)f(x,u) + \frac{1}{2}V_{xx}(t,T,x)\sigma^2(x,u)\right\}, \\ (T,T,x) = 0. \end{gathered} \tag{9}$$

We explicitly include the parameter T, which indicates the length of the finite time horizon, in the notation and consider the value function as a function of three variables. For numerical and analytical results, a fundamental question is whether the value functions $V(t,T,x)$ represent a reasonable approximation for the value function $V(x)$ of the infinite horizon approximation problem.

Proposition 2. Assume that the value function V(x) of the infinite horizon problem exists and satisfies (TVC). Then

$$\mathrm{Lim}_{T\to\infty} V(t,T,x) = V(x).$$

Independent of $t \in [0, \infty)$ and for all x.

Proposition 3. Assume that the value functions of the finite T-horizon problems can be written as

$$V(t,T,x) = A_1(t,T)g_1(x) + A_2(t,T)g_2(x) + \ldots + A_n(t,T)g_n(x).$$

Satisfying the following conditions:

I. The functions gi(x) are linearly independent (As functions).

II. For all I, the following limit exists.

$$\mathrm{Lim}_{T\to\infty} A_i(t,T) = A_i.$$

III. For all i and admissible controls.

$$\mathrm{Lim}_{t\to\infty} E(e^{-rt}\left|g_i\big(x(t)\big)\right|) = 0.$$

Then, the value function of the infinite horizon problem exists and is given by

$$V(x) = A_1g_1(x) + A_2g_2(x) + \ldots + A_ng_n(x). \tag{10}$$

4.1 | A New Hybrid Method

In Hesameddini et al. [13], MSDM is introduced and used for solving the finite horizon SOCPs. In this section, we propose a new variation of this approach introduced in Section 3 for solving infinite horizon SOCPs. To be sure, solving governed equations and making a sophisticated guess for value function are major weaknesses of the method introduced in Section 4. In this article, these objections are elevated with a stochastic differential transformation. In this paper, the novelty comes from the fact that:

I. We hybridize the methods for solving infinite-horizon SOCPs, described in Section 4, with one of the effective methods for solving differential equations, SDTM.

II. Similar to Villafuerte et al. [14], the DTM is developed to solve stochastic differential equations, but in our new method, SDTM is applied to definite horizons.

III. Our new approach complements the approximation method to obtain an analytic-numerical approximation of optimal control and trajectory.

IV. We assume that *Model (1)* 's parameters belong to L4-space.

V. Due to the special spaces in *Model (4)* , the utility function is selected in HARA1 types. This topic eliminates the problem of constructing a value function in a truncated definite problem.

However, we write all the steps for determining optimal control and trajectory step by step. The algorithm that will be used to obtain these purposes for *Model (1)* is as follows:

Algorithm

Step 1. For *Model (1)*, write the truncated *Model (7)*;

Step 2. Set HJB-equation for *Model (7)* and suppose that for each T, the function V (., T,.) defined on $[0, T] \times S$, is of class $C^{1,2}$ and satisfies in *Model (9)*.

Step 3. Make a sophisticated guess for the value function as *Model (10)* to obtain V(x);

Step 4. Compute the optimal control in feedback form from maximizing the right-hand side of *Model (9)*;

Step 5. Solve SDE with SDTM by substituting optimal control in the governed system equation.

The answer obtained with the above algorithm is according to variable T. Perhaps this topic is the most important advantage of our new method.

4.2 | A Case-Study and Simulation: Merton's Portfolio Problem

To explain the application of the mentioned method for solving SOCPs, as an example, we solve the well-known economic problem that Merton works out in [5]. Merton's portfolio problem is famous in continuous-time finance. This problem was formulated and solved by Merton [17] in 1969 for both finite lifetimes and infinite time horizons. A portfolio's optimization problem mainly involves describing an individual's investment choices whose degree of risk aversion is known, which is defined by the utility function. In its first version, the model assumes a market with a certain number of risky securities and risk-free security. The individual has an initial wealth to invest freely in the available assets. The objective function to maximize is given by the function of the expected utility of the wealth to a certain future moment that represents the temporal horizon of the optimization problem. In Merton's [17] portfolio problem, investors can invest only in stock assets. In the discussion section of this paper, we have tried to understand what changes should be made to the solution if the investor has the opportunity to access both the stock market and the derivatives market [18], [19].

In Merton's [17] model, we can invest part of our wealth in either a risk-free bond or a risky stock while also planning to consume a portion of our wealth as time progresses. This problem assumes that an investor holds a portfolio of two assets: A risk-free bond and risky stock. The price $b(t)$ per share of the bond evolves according to $db = rbdt$. In contrast, the price of the stock follows the stochastic differential equation $dS = S(Rdt + \sigma dB)$, where r, R σ and are constants, and $R > r > 0$, $\sigma \neq 0$. Let $x(t)$ denote the investor's wealth at t t he time, $u_1(t)$ be the fraction of wealth allocated to the risky asset, and $u_2(t)$ represent the consumption rate. Thus, $u(t) = (u_1(t), u_2(t))$ is control variable and attains its values in $U = [0,1] \times [0, +\infty)$. In this manner, the total wealth evolves as follows:

$$dx(t) = \left(1 - u_1(t)\right)x(t)\frac{db}{b} + u_1(t)x(t)\frac{dS}{S} - u_2(t)dt,$$

therefore,

$$dx(t) = r\left(1 - u_1(t)\right)x(t)dt + u_1(t)x(t)\{Rdt + \sigma dB(t)\} - u_2(t)dt, x(0) = x_0. \quad \textbf{(11)}$$

We stop the process if the wealth reaches zero (bankruptcy). In addition, we assume that the running cost is $F(x(t), u(t)) = l(u_2(t))$, where $l(u_2(t))$ represents the utility of consuming at the rate $u_2(t) > 0$. Of note, the problem is to maximize the total expected utility, discounted at rate $r > 0$:

$$E(\int_0^\tau e^{-rt} l(u_2(s))ds),$$

where τ denotes the random first time $x(.)$ leaves $Q = \{(x,t): 0 \leq t \leq T, x \geq 0\}$.

For this problem, we apply the algorithm described in the previous section. In *Step 1*, we have the truncated problem and choose the following utility function, which is of the hyperbolic absolute risk aversion type [2], [20]. In the next step, the HJB equation takes the form of *Eq. (11)*.

In Krawczyk [7], the HJB equation for this SOCP is given as:

$$v_t + \max_{\substack{u_2 \geq 0 \\ 0 \leq u_1 \leq 1}} \{\frac{(u_1 x\sigma)^2}{2} v_{xx} + (r(1 - u_1)x + Rxu_1 - u_2)v_x + e^{-\beta t} l(u_2)\} = 0, \quad \textbf{(11)}$$

With boundary conditions

$$v(0,t) = 0 \text{ and } v(x,T) = 0.$$

The chosen utility function, which is of the hyperbolic absolute risk aversion type [2], [20] is:

$$l(c) = \frac{1}{\gamma} c^\gamma; \;\; 0 < \gamma < 1.$$

For the next step, the optimal control is computed by maximizing the expression in the HJB equation. This yields the following optimal control form:

$$U^* = (U_1^*, U_2^*) = \left(-\frac{(R - r)V_x}{\sigma^2 x V_{xx}}, V_x^{\frac{1}{\gamma - 1}}\right).$$

It is assumed that an investor with an initial wealth of $x_0 = 10^5$ units aims to maximize their satisfaction over the next ten years. The parameters considered in this study are $r = 0.05, R = 0.11, \sigma = 0.4$, and $\gamma = 0.5$. This paper assumes that the investor can invest using the above data over an infinite horizon. Substituting the derived optimal control into *Eq. (12)*, we obtain the following equation:

$$\beta V - V_t = \frac{1}{V_x} + rxV_x - \frac{1}{2}\frac{(R - r)^2 V_x^2}{\sigma^2 V_{xx}}; \;\; V(T,T,x) = 0. \quad \textbf{(13)}$$

To solve *Eq. (13)*, we define the following value function as:

$$V(t,T,x) = A_1(t,T)\sqrt{x}.$$

Given the boundary condition of *Eq. (13)*, this holds in state space for all x, implying that:

$$A_1(T,T) = 0. \tag{14}$$

By substituting $A_1(T, T)=0$ from *Eq. (14)* into *Eq. (13)*, we obtain the following differential equation, which is of Bernoulli equation type:

$$\dot{A}_1(t,T) - \left(\beta - 2r - \frac{(R-r)^2}{2\sigma^2}\right)A_1(t,T) = \frac{-2}{A_1(t,T)};\ A_1(T,T) = 0.$$

By solving the differential equation, we obtain the following solution for $A_1(t, T)$:

$$A_1(t,T) = e^{\theta t}\sqrt{\frac{2}{\theta}(e^{-2\theta t} - e^{-2\theta T})}.$$

The corresponding value function is:

$$V(t,T,x) = e^{\theta t}\sqrt{\frac{2x}{\theta}(e^{-2\theta t} - e^{-2\theta T})},$$

where

$$\theta = \beta - 2r - \frac{(R-r)^2}{2\sigma^2}.$$

It can then be included from the above result that

$$\lim_{T\to\infty} A_1(t,T) = \sqrt{\frac{2}{\theta}}.$$

Therefore, the suitable candidate for the value function is:

$$V(x) = \sqrt{\frac{2x}{\theta}}.$$

We, therefore, obtain that the optimal control is

$$U^* = (U_1^*, U_2^*) = \left(\frac{2(R-r)}{\sigma^2}, 2\theta x(t)\right).$$

By substituting this form of optimal control into governed *Eq. (11)*, we *Eq. (15)*:

$$dx(t) = \left(5r + 3\frac{(R-r)}{\sigma^2} - 2\beta\right)x(t)dt + \frac{2(R-r)}{\sigma^2}x(t)dB(t). \tag{15}$$

Which describes the dynamics of the investor's wealth over time.

By solving the above equation using SDTM for a definite horizon in the mean square space, we can obtain simulations of the optimal control and trajectory as a function of T. *Fig. 1* presents the approximated optimal controls for 5 random runs, assuming T=1o. We examine the solution obtained with the hybrid approach and substitute one instead of T. The new method was applied to this problem in the second stage with fifty random runs, assuming T=10.

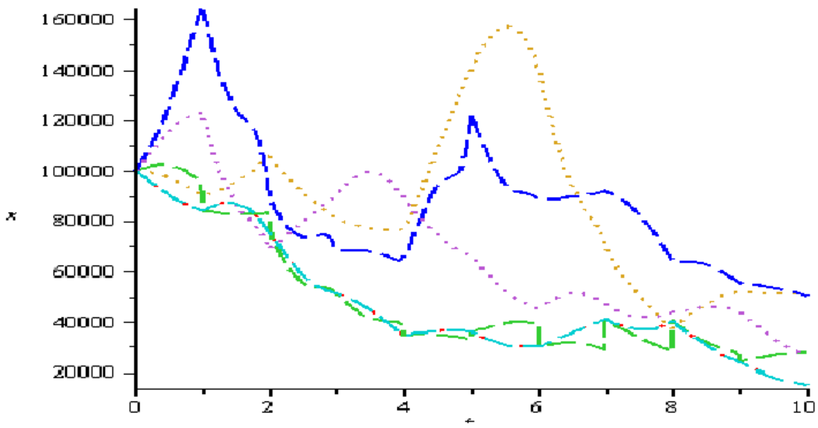

Fig. 1. Five samples of trajectory.

The average wealth from these simulations is plotted in *Fig. 2*.

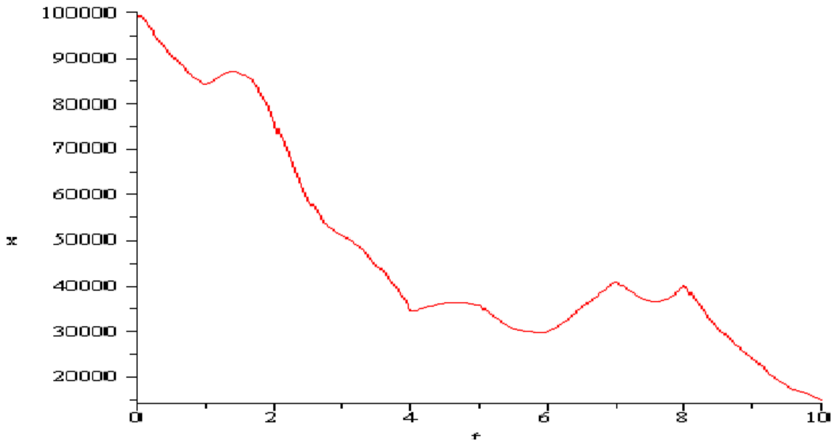

Fig. 2. The average wealth is running fifty times randomly.

For this choice, the obtained solution is compared with the results from [21]. Our results have been satisfactory for several reasons. First, the obtained optimal trajectory matches the result Kirk [21] presented, indicating that the introduced approach is valid and reliable. Second, we introduce a time continues trajectory and optimal control strategy. Third, selecting the endpoint in this problem implies determining the optimal investment strategy that aligns with when the investment period concludes. Accordingly, the value of x reaches zero at this time. This fact is consistent with the results of Gokdogan and Merdan [11] using the new method. The endpoint or selection of a definite horizon can represent the equilibrium point of the system. Therefore, accurately estimating this point demonstrates the efficiency and effectiveness of the new approach.

5 | Conclusion

This work presents a new method for solving infinite-horizon SOCPs to determine an approximate analytical strategy. The obtained solution is a series in which the coefficients can be computed accurately. Our hybrid approach has advantages over previous methods, such as the variational iteration method and Markov chain techniques. First, it provides a continuous curve for the solution; second, the convergence of the new approach is guaranteed, and the solution is presented as a polynomial with suitable approximation. An advantage of this approach is that it provides solutions for SOCPs while accounting for noise in the system. This topic is particularly useful in addressing financial and economic problems, such as nominal, real, and productivity shocks. It is designed to outline and evaluate various strategies to manage and mitigate such challenges effectively. We want to emphasize that the novelty presented in this paper lies in the hybridization of two well-known methods, effectively creating a comprehensive approximation approach.

Conflict of Interest

The authors declare that there is no conflict of interest regarding the publication of this article.

Data Availability

All data generated or analyzed during this study are included in this published article. No additional data are available.

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